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## TABLE OF CONTENTS

**Foundations** . . . . . 635

**Algebra** . . . . . 637
Abstract algebra . . . . . 638

**Theory of groups** . . . . . 639

**Number theory** . . . . . 640

**Analysis** . . . . . 646
Theory of sets, theory of functions of real variables . . . . . 646
Theory of functions of complex variables . 648
Theory of series . . . . . 653
Fourier series and generalizations, integral transforms . . . . . 654
Polynomials, polynomial approximations . 661
Special functions . . . . . 662
Harmonic functions, potential theory . . 663
Differential equations . . . . . 664
Functional analysis, ergodic theory . . . 668
Theory of probability . . . . . 671
Mathematical statistics . . . . . 672

**Topology** . . . . . 675

**Geometry** . . . . . 678
Convex domains, integral geometry, extremal problems . . . . . 680
Algebraic geometry . . . . . 682
Differential geometry . . . . . 686

**Numerical and graphical methods** . . . 691

**Astronomy** . . . . . 694

**Mechanics** . . . . . 696
Hydrodynamics, aerodynamics, acoustics . 696
Elasticity, plasticity . . . . . 700

**Mathematical physics** . . . . . 704
Optics, electromagnetic theory . . . . . 704
Quantum mechanics . . . . . 706

# AUTHOR INDEX


Abramowitz, M. 691
Agmon, S. See Mandelbrojt, S.
Aizenštat, N. D. 664
Aleksandrov, A. D. 680
Aleksandrov, A. D.-Pogorelov, A. V. 680
Aleksandrov, P.-Sitnikov, K. 676
Alexits, G. 659
Allen, D. N. de G.-Southwell, R. 703
Allendoerfer, C. B. 689
Al'perin, M. M. 706
Alumyaè, N. A. 701
Andersen, E. 693
Apostol, T. M. 641
Arutyunyan, N. H. 702
Arvesen, O. P. 685
Backes, F. 687
Bader, R. See Germain, P.
Bagchi, H. 682
Bar-Hillel, Y. 635
Barnard, G. A. 672
Barrois, W.-Simon-Suisse, J. 702
Barsotti, I. 639
Bartlett, M. S. 671, 672
Basile, R. 692
Bass, J. 699
Bassière, M. See Tournier, M.
Batchelor, G. K. 699
Bauer, F. L. 636
Beck, E. 693
Begle, E. G. 677
Behnke, H.-Sommer, F. 652
Behnke, H.-Stein, K. 652
Belardinelli, G. 649
Bell, J. C. 689
Berman, M. È. 700
Bilharz, H. 699
Binnie, A. M.-Harris, D. P. 698
Birkhoff, G. 638, 640
Birnbaum, Z. W. 673
Bjorna, S. J. 674
Boas, R. P., Jr. 655
Bochner, S.-Chandrasekharan, K. 646, 659
Bode, H. W.-Shannon, C. E. 672
Bol, G. 689
Bompiani, E. 688, 689
Bondi, C. M.-Bondi, H. 695
Bondi, H. See Bondi, C. M.
Bonferroni, C. 679
Booth, A. D. 693
Borel, A. 640
Brändli, E. R.-Eckmann, B. 677
Brock, J. E. 666
Brodin, J. 706
Brown, B.-McCoy, N. H. 638
Bruwier, L. 664
Buchdahl, H. A. 694
Buchholz, H. 663, 705
Cartan, H.-Leray, J. 678
Carter, A. H. 673
Cassels, J. W. S. 643
Castellani, M. 673
de Castro Brzezicki, A. 649
Casulleras, J. 682
Cetlin, M. L. See Gel'fand, I. M.
Chandrasekharan, K. See Bochner, S.
Chastenet de Géry, J. See Souriau, J.-M.
Cheng, Min-Teh 659
Chiellini, A. 665
Choquet, G. 647
Cicala, P. 702
Citlanadze, È. S. 670
Collatz, L. 692
Conti, R. 668
Cossu, A. 688
Cotton, É. 663
Császár, Á. 679
Cuesta, N. 646
Dean, W. R. 697
Dedò, M. 686
Delachet, A. 691
Delgleize, A. 674
Del Pasqua, D. 668
Denjoy, A. 647
Denk, F.-Haupt, O. 690
Dieudonné, J.-Gomes, A. 668
Diliberto, S. P. 665
Dilworth, R. P. 647
Doss, R. 659
Dowker, Y. N. 671
Dreyer, H. J. 693
van den Driessche, R. 636
Dubois-Violette, P.-L. 666
Dvoretzky, A.-Erdös, P.-Kakutani, S. 671
Džrbašyan, M. M. 649
Eckmann, B. See Brändli, E. R.
Egerváry, E. 650
Ehresmann, C. 678
Eisenbud, L. See Wigner, E. P.
Eisenhart, L. P. 687
Elianu, I. P. 668
El'sgol'c, L. È. 671
Epheser, H. 649
Erdös, P. 642
Erdös, P. See Dvoretzky, A.
Erdös, P.-Rényi, A. 644
Erugin, N. P. 667
Everett, C. J.-Ryser, H. J. 668
Faedo, S. 636
Faragó, P. S.-Takács, L. 672
Favard, J. 669
Federer, W. T. 674
Federhofer, K. 702
Feller, W. 674
Fichera, G. 663, 700
Figueras Calsina, E. 675
Floyd, E. E. 676
Fáry, E. 640, 659
Fränz, K. 660
Freudenthal, H. 686
Fuchs, L. 638
Gál, I. S. 659
Garnir, H. 667
Gatteschi, L. 662
Gel'fand, I. M.-Cetlin, M. L. 639
Geller, L. See Wiener, N.
Gerber, R. 696
Germain, P.-Bader, R. 667
Germay, R. H. J. 664
Ghidoli, G. B. 672
Ghizzetti, A. 667
Ghosh, M. N. 671
Gilbarg, D.-Serrin, J. 696
Gnedenko, B. V. 637
Godeaux, L. 685
Godwin, H. J. 673
Gołąb, S. 690
Gołąb, S. See Hoborski, A.
Gomes, A. See Dieudonné, J.
Goodman, T. R. 699
Goodspeed, F. M. 661
Goodstein, R. L. 636
Gosse, R. 665
Gotô, K. 639
Gougenheim, A. 679
Green, A. E.-Zerna, W. 701
Greville, T. N. E. 692
Grioli, G. See Tolotti, C.
Gröbner, W. 638
Grundy, P. M. 674
Guest, P. G. 692
Haack, W. 687
Hadwiger, H. 680
Harris, D. P. See Binnie, A. M.
Haupt, O. 682


(*Continued on cover 4*)

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# Mathematical Reviews

Vol. 11, No. 9 OCTOBER, 1950 Pages 635-706


## FOUNDATIONS

**Bar-Hillel, Yehoshua. On syntactical categories.** J. Symbolic Logic 15, 1–16 (1950).

Verfasser erörtert die Aufgabe, die Umgangssprache durch Kalküle zu approximieren. Dazu ist es nötig, auf eine einfache Syntax, nach der für jeder prädikat $P$ und jeder Individuum $i$ stets $P(i)$ eine Aussage ist, zur verzichten. Es wird an Beispielen gezeigt, welche Komplikationen die Umgangssprache statt dessen enthält. Zur Untersuchung solcher Syntax wird definiert: (1) Zwei Ausdrücke $e_1$, $e_2$ heissen "isogen," wenn aus jeder Aussage bei Ersetzung von $e_1$ durch $e_2$ (oder umgekehrt) wieder eine Aussage entsteht; (2) $e_1$, $e_2$ heissen "verwandt," wenn es zwei Aussagen gibt, die bei Ersetzung von $e_1$ durch $e_2$ auseinander entstehen; (3) $e_1$, $e_2$ heissen "versippt," wenn es eine Verwandtschaftskette von $e_1$ nach $e_2$ gibt. "Isogen" und "versippt" sind Äquivalenzrelationen. *P. Lorenzen* (Bonn).

**Martin, R. M. A note on nominalistic syntax.** J. Symbolic Logic 14, 226–227 (1950).

The author describes a different and more economical way of deriving nominalistic syntax in the system of Goodman and Quine [same J. 12, 105–122 (1947); these Rev. 9, 262], in terms of a relation $L$ meaning "is to the left of." *O. Frink* (State College, Pa.).

**Słupecki, Jerzy. On proper rules of inference.** Kwartalnik Filozoficzny 18, 309–312, 325–326 (1949). (Polish. English summary)

Simplification of the system of proper rules of inference given by Suszko [same Kwartalnik 17, 199–205, 319–320 (1948); these Rev. 10, 421] (a rule is proper when in it no premiss nor conclusion is tautologically true) and a general theorem concerning elimination of axioms of propositional calculus by proper rules. If $\alpha_1, \cdots, \alpha_n$ is a system of axioms in none of which appears the variable $s$, then for their elimination it is sufficient to accept the following $n+1$ rules: (1) $Ksp \to p$; (2) $s \to Ks\alpha_i$, $i = 1, 2, \cdots, n$. *J. Łoś.*

**Kuznecov, A. V. On primitive recursive functions of large oscillation.** Doklady Akad. Nauk SSSR (N.S.) 71, 233–236 (1950). (Russian)

A primitive recursive function $g(x)$ is called of large oscillation if it has the property $(x)(y)(Ez)z > x \ \& \ g(z) = y$. [Throughout this review all variables range over the natural numbers.] It was shown by Markov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 417–424 (1949); these Rev. 11, 151] that if $g(x)$ is such a function, then an arbitrary general recursive function $f(x_1, \cdots, x_n)$ can be represented in the form $f(x_1, \cdots, x_n) = g\{\mu y[\phi(x_1, \cdots, x_n y) = 0]\}$ where $\phi(x_1, \cdots, x_n)$ is primitive recursive. This paper announces some new results concerning such functions in relation to primitive and complex (i.e., nonprimitive) recursive functions. For any function $f(x)$, let $f'(x)$ be the number of $i < x$ such that $f(i) = f(x)$. If $g(x)$ is of large oscillation, then $\vartheta(x, y) = (\mu z)[g'(z) = x \ \& \ g(z) = y]$ is defined for all $x, y$ and establishes a one-to-one correspondence between the pairs of numbers and numbers. The same thing can be done for $n$-tuples by an easy iteration; and for arbitrary finite sets of integers if $g(0) = 0$ and $x = 0 \to g(x) > x$. The author attacks the question of whether there exists a primitive recursive $g(x)$ of large oscillation such that $\vartheta(x, y)$ is not primitive recursive. This reduces to the question of whether there is a primitive recursive one-to-one mapping of the natural numbers on themselves such that the inverse is not primitive recursive. This in turn reduces to the question of whether there exists a complex recursive function $\phi(x)$, such that the predicate $(Et)\ (\phi(t) = x)$ is primitive recursive; such functions $\phi(x)$ constitute the class $R$. The author answers all three questions in the affirmative by exhibiting a function $F(x, y)$, defined by $F(x, 0) = 2x$, $F(0, y+1) = 1$, $F(x+1, y+1) = F(F(x, y+1), y)$, for which $f(x) = F(x, x)$ is in the class $R$. He further asserts that if $\phi(x)$ is complex recursive and if the number of substitutions necessary to calculate $\phi(x)$ does not exceed $\psi(\phi(x))$, where $\psi$ is primitive recursive, then $\phi(x)$ belongs to $R$; that every general recursive function is the difference of two functions belonging to $R$; and finally that given any general recursive $f(x)$ and primitive recursive $g(x)$ of large oscillation, there exists a primitive recursive $g_1(x)$ of large oscillation such that $f(x) = g(\vartheta_1(0, x+1))$. *H. B. Curry* (Louvain).

**Skolem, Th. A remark on the induction scheme.** Norske Vid. Selsk. Forh., Trondheim 22, no. 36, 167–170 (1950).

The author shows that the induction scheme

$$\frac{A(0) \quad A(n) \to A(n')}{A(n)}$$

in recursive arithmetic is equivalent to either of the schemes

$$S_1 \qquad \frac{f(n) \leqq f(n')}{f(0) \leqq f(n)},$$

$$S_2 \qquad \frac{f(n) \geqq f(n')}{f(0) \geqq f(n)},$$

$$S_3 \qquad \frac{f(0) = 0 \quad f(n) \geqq f(n')}{f(n) = 0},$$

provided that we assume for the characteristic function $\alpha(n)$, corresponding to $A(x)$, the formula $\alpha(n) = 0 \vee \alpha(n) = 1$, and presuppose certain properties of some particular elementary functions. As for the scheme

$$S_4 \qquad \frac{f(0) = 0 \quad f(n) = f(n')}{f(n) = 0}$$

the question is left open. [The notation here is that of the Hilbert school. "$A(n)$" represents the statement that the property $A$ holds for $n$, while "$f(n)$" represents the value, for $x = n$, of the numerical function $f(x)$.] *H. B. Curry.*

**Wang, Hao. A formal system of logic.** J. Symbolic Logic **15**, 25–32 (1950).

The author proposes a system $P$ which differs from that of Quine's book [Mathematical Logic, Norton, New York, 1940; these Rev. **2**, 65] in having a slightly weaker elementhood axiom to replace Quine's *200, which is too strong. This new axiom requires that the class determined by a stratified formula $\phi$ be an element only if $\phi$ is normal, which means that all the bound variables in $\phi$ are element variables. Rosser's derivation of the Burali-Forti paradox in the Quine system fails here, and the author gives reasons why modifications of Rosser's derivation may also be expected to fail. It is shown that the system $P$ is consistent if Quine's earlier and weaker system [Amer. Math. Monthly **44**, 70–80 (1937)] is consistent. A model satisfying the axioms of $P$ is constructed, whose elements are classes of natural numbers. All the theorems of Quine's book may be derived in $P$. In short, this system has the advantages of the system of Quine's Mathematical Logic, plus the additional advantage that its consistency, though not proved, is made to seem plausible. *O. Frink* (State College, Pa.).

**Bauer, F. L. Zur Algebraik des Logikkalküls.** Methodos **1**, 288–292 (1949).

This is an elementary exposition of the classical sentential calculus, which is here regarded as the theory of polynomials over a two-element field. *J. C. C. McKinsey.*

**Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. I.** Nederl. Akad. Wetensch., Proc. **53**, 327–336 = Indagationes Math. **12**, 75–84 (1950).

Two subsystems of intuitionistic propositional calculus and systems dual to these are described. The simplest, $A_1$, adds a propositional constant (to be interpreted as a true proposition) to the calculus for positive identical $I$–$K$ formulas [Hilbert and Bernays, Grundlagen der Mathematik, v. 2, Springer, Berlin, 1939]. The system $A_2$, which may be interpreted as a distributive lattice, contains in addition a symbol for disjunction. The characterization of both systems, similar to the author's for classical propositional logic [Nederl. Akad. Wetensch., Proc. **49**, 1153–1164 = Indagationes Math. **8**, 701–712 (1946); these Rev. **8**, 306], is obtained by adding substitution rules to axioms and rules with a direct lattice theoretic interpretation. In a continuation the author will extend these systems to obtain the minimal and intuitionistic calculi and corresponding dual systems. *D. Nelson* (Washington, D. C.).

**van den Driessche, René. Sur le "De syllogismo hypothetico" de Boèce.** Methodos **1**, 293–307 (1949).

Zur Interpretation der logischen Arbeiten von Boethius [vgl. K. Dürr, Erkenntnis **7**, 160–168 (1938)] gibt Verf. in gewissen Thesen von Boethius der lateinischen Partikel "si" nicht die übliche Bedeutung der Implikation [$\subset$ bei Łukasiewicz], sondern der Äquivalenz $E$. In Übereinstimmung mit dem Kommentar von Abälard wird dem (meist mit "si" gleichbedeutenden) "cum" gelegentlich die Bedeutung der Konjunktion $K$ gegeben. Schliesslich wird noch für "aut" in einigen Thesen die Bedeutung von $NE$ vorgeschlagen. Boethius hätte nach dieser Hypothese keine falschen Aussagen für wahr gehalten, er hätte sich nur uneindeutigen Sprachgebrauch unschulden kommen lassen. *P. Lorenzen* (Bonn).

**Goodstein, R. L. The Gauss test for relative convergence.** Amer. J. Math. **72**, 217–228 (1950).

The recursive theory of convergence differs from the ordinary theory in that (roughly speaking), whenever the latter says "for each $\epsilon>0$ there exists an $n$," the $\epsilon$ is taken to be $1/k$, where $k$ is a positive integer, and the $n$ is required to be a recursive function of $k$. The notion of recursiveness intended here is the general recursiveness developed by Kleene [see, e.g., Math. Ann. **112**, 727–742 (1936)] following suggestions of Gödel and Herbrand. Since this notion is a kind of effectiveness, the recursive theory of convergence is essentially a theory of effective convergence which has significance from an intuitionist (or other constructivist) standpoint. The present paper is a contribution to this field. It deals with a relative convergence introduced by the author in a previous paper [Proc. London Math. Soc. (2) **48**, 401–434 (1945); these Rev. **8**, 245]; a sequence $f(m, n)$ is (roughly speaking) convergent in $n$ relative to $m$ if it satisfies the Cauchy condition for convergence in $n$ only for sufficiently large values of $m$. The objective is to establish the validity of a test for this sort of convergence which is analogous to the Gauss test for ordinary positive-term series. In the course of doing this the author establishes analogues of the comparison tests, Cauchy's radical test, d'Alembert's test, the original form of Kummer's test, and Raabe's tests, first for ordinary and then for relative convergence. He shows furthermore that the classical result which asserts the convergence of a bounded monotone sequence does not hold in the recursive theory, so that the above results have to be obtained without it. There is also a rather technical generalization. *H. B. Curry.*

**Faedo, Sandro. Chiarimento a una nota del professore W. Sierpiński. I.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 572–576 (1949).

**Faedo, Sandro. Chiarimento a una nota del professore W. Sierpiński. II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 667–672 (1949).

In these notes the author clarifies the relation between a previous note by himself [Ann. Scuola Norm. Super. Pisa (2) **10**, 209–214 (1941); these Rev. **8**, 449] and a note by Sierpiński [same Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 216–217 (1947); these Rev. **9**, 573]. In Sierpiński's note it was shown that if a given countable set is dense in a complete metric space $M$, a rule can be specified for selecting an element from each nonnull closed subset of $M$. Here Faedo applies this result to four spaces of continuous functions. In the second note he gives another way of specifying the selection rule for these spaces. In Sierpiński's note it was shown that if a rule were available for selecting an element from each closed subset of the space of bounded continuous functions on the real axis (in the usual metric), it would lead to a rule for well-ordering the real numbers. But this space is not separable, and is not one of those considered by Faedo. *L. M. Graves* (Chicago, Ill.).

**Riabouchinsky, Dimitri. Nouvelles remarques sur le problème de la règle des signes.** C. R. Acad. Sci. Paris **230**, 421–424 (1950).

Further remarks on the author's new rule of signs [same C. R. **229**, 405–408 (1949); these Rev. **11**, 152]. This rule is consistent with addition and subtraction. The imaginary unit $\sqrt{-1}$ has a new interpretation. A symmetry theorem is proved. *O. Frink* (State College, Pa.).

**Gnedenko, B. V. The theory of probability and knowledge of the real world.** Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 3–23 (1950). (Russian)

This is a politico-philosophical discussion of probability theory. The author makes several references to Marx, Engels, Lenin, Stalin, and the biologist Lysenko. (The latter is quoted as saying, among other things, that science is the enemy of chance.) It is asserted that bourgeois scientists study nature not because such a study is useful, but for the intellectual pleasure it yields: a point of view abhorrent to the author. One of the main points of the paper is the assertion that a mathematician solving a practical problem is responsible not only for the correctness of his deductions, but also for the formulation of the problem. If, in other words, an engineer asks a mathematician for help in solving a differential equation, it is part of the mathematician's duty to verify that he is being asked to solve the right equation. The author regrets the fact that, frequently in the past, Soviet scientists were carried away by their interest in the simplicity of a problem and in the ingenuity of its solution, without paying enough attention to its practical value. As an application of this principle, he expresses remorse at having used Mendel's laws as an illustrative example in his book on probability (written jointly with Khintchine), and he chides his colleagues Glivenko, Kolmogoroff, and Liapounoff for having devoted a few articles to formal genetics and, in particular, to a "proof" of Mendel's laws.

*P. R. Halmos* (Chicago, Ill.).

**Turnbull, H. W. Commentary: Mathematics in the larger context.** Research **3**, 197–201 (1950).

## ALGEBRA

**Kishen, K. On the construction of latin and hyper-graeco-latin cubes and hypercubes.** J. Indian Soc. Agric. Statistics **2**, 20–48 (1949).

An $m$-fold $s$-sided Latin cube of the first order is defined as an arrangement of $s$ letters each repeated $s^{m-1}$ times into the points of an $m$-dimensional cube of side $s$, such that each letter occurs exactly $s^{m-2}$ times on every $(m-1)$-dimensional hyperplane parallel to the coordinate hyperplanes. Two such cubes are called orthogonal if each letter of one occurs exactly $s^{m-2}$ times with each letter of the other. For instance, if a set of $k$ orthogonal squares is repeated in all the planes parallel to $(OX_1X_2)$, a set of $k$ orthogonal hypercubes of first order is obtained. A set of $(s^m-1)/(s-1)-m$ orthogonal $m$-fold hypercubes of side $s$ is called an $s$-sided $m$-fold completely orthogonalized hyper-Graeco-Latin-hypercube of the first order. An $s$-sided $m$-fold Latin hypercube of the $r$th order $(r<m)$ is defined as an arrangement of $s^r$ letters each repeated $s^{m-r}$ times such that each letter occurs exactly $s^{m-r-1}$ times in each $(m-1)$-dimensional hyperplane parallel to the coordinate hyperplanes. Two such Latin hypercubes are called orthogonal if every letter of one occurs $s^{m-2r}$ times with every letter of the other.

By an extension of the methods used by R. C. Bose [Sankhyā **3**, 323–338 (1938)] the author constructs for any $m$ an $m$-fold completely orthogonalized hyper-Graeco-Latin-hypercube of the first order and of side $s$ when $s$ is the power of a prime. It is important to note that for $m>2$ a finite geometry cannot necessarily be constructed from the configuration. Among the Latin hypercubes composing the orthogonalized configuration there are $(s-1)^{m-1}$ with the property that every plane parallel to any coordinate plane represents a Latin square. The author also describes a method of obtaining $m$-fold Latin hypercubes of order $r$. This method also uses the properties of finite projective geometries. Several examples illustrating the author's methods are presented.

*H. B. Mann* (Berkeley, Calif.).

**Szmydtówna, Z. Sur les racines caractéristiques et sur les directions caractéristiques de certaines matrices.** Ann. Soc. Polon. Math. **22** (1949), 235–240 (1950).

Several theorems due to Frobenius [S.-B. Preuss. Akad. Wiss. **1908**, 471–476; **1909**, 514–518] concerning matrices with positive or nonnegative elements are generalised to the cases of matrices whose off-diagonal elements are positive ($a^+$-matrices) or nonnegative ($a$-matrices). The author observes that the original proofs of some of these theorems made no use of the signs of the main diagonal elements. For other theorems the proof can be reduced to the Frobenius case. It is shown that the characteristic root with largest real part of an $a$-matrix is real [see also Rohrbach, Jber. Deutsch. Math. Verein **40**, 49–53 (1931)]; for an $a^+$-matrix this root is simple. The components $x_i$ of the corresponding characteristic vectors are such that $x_ix_j \geqq 0$ for an $a$-matrix and $x_ix_j>0$ for an $a^+$-matrix. The components of a vector which corresponds to a smaller real characteristic root cannot all be positive. Finally, a theorem concerning the vectors which correspond to complex roots is proved; this result has no analogue among the Frobenius theorems.

*O. Todd Taussky* (Washington, D. C.).

**Potter, H. S. A. On the latent roots of quasi-commutative matrices.** Amer. Math. Monthly **57**, 321–322 (1950).

The author considers matrices $A, B$ of order $n$ such that $AB=\omega BA$, where $\omega$ is a $q$th root of unity. He proves that the latent roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ of $A$ and $\beta_1, \beta_2, \cdots, \beta_n$ of $B$ may be so arranged that the latent roots of $A+B$ are $(\alpha_r^q+\beta_r^q)^{1/q}$, $r=1, 2, \cdots, n$, and those of $AB$ are $\omega^{\frac{1}{2}q(q-1)}\alpha_r\beta_r$, $r=1, 2, \cdots, n$. [Reviewer's note. The values of the fractional powers occurring in the statement are not determined, and different branches of the many-valued functions may have to be chosen for different latent roots.]

*W. Ledermann* (Manchester).

**Vinograde, Bernard. Canonical positive definite matrices under internal linear transformations.** Proc. Amer. Math. Soc. **1**, 159–161 (1950).

All matrices considered are square matrices of order $n$ with real elements. Let $P$ be a symmetric positive definite matrix, and suppose that $n=p+q+r$ where $p \leqq q \leqq r$ (actually, $n$ may be similarly partitioned into any number of parts). If $S=\text{diag}(S_1, S_2, S_3)$, where $S_1$, $S_2$, and $S_3$ are nonsingular square matrices of orders $p$, $q$, and $r$, respectively, then a transformation $P \to SPS'$ is called an internal linear transformation. The author discusses a canonical form for $P$ under internal linear transformations, the canonical form depending on systems of characteristic values associated with $P$. Applications to statistics are suggested.

*N. H. McCoy* (Northampton, Mass.).

**Parodi, Maurice. Sur une application d'un théorème de M. Müller.** Bull. Sci. Math. (2) **73**, 192–196 (1949).

It is known that a real matrix $(a_{ik})$ with $a_{ii}>\sum_{i\neq k}|a_{ik}|$ is positive definite. Using a theorem of Müller [Math. Z. **51**,

291–293 (1948); these Rev. 10, 278] the author constructs a larger class of positive definite matrices.

*O. Todd Taussky* (Washington, D. C.).

**Sircar, H. On a system of equations.** Proc. Edinburgh Math. Soc. (2) 8, 101–105 (1949).

Let **A** be a real symmetric matrix of order $n>2$ and rank $n-1$; let **u** and **l** be column vectors of order $n$; let $\mathbf{l}'$ be the transpose of **l**, and let $p$ be a scalar. Then the $n+2$ equations obtained by equating to zero all the $(n+1)$-rowed minors of the matrix

$$\begin{Vmatrix} \mathbf{A} & \mathbf{u} & \mathbf{l} \\ \mathbf{l}' & p & 0 \end{Vmatrix}$$

are equivalent to only two equations. One of these is linear in the components of **l**, and the other is a quadratic form in certain $n-1$ components of **l**. *W. Ledermann.*

**Schwerdtfeger, H. On the Pfaffian invariant of a skew-symmetric matrix.** Proc. Edinburgh Math. Soc. (2) 8, 106–110 (1949).

Every nonsingular skew-symmetric matrix $P$ of order $2m$ can be expressed, in many ways, in the form

$$P=[u_{(1)}, u_{(2)}]+\cdots+[u_{(2m-1)}, u_{(2m)}],$$

where $u_{(1)}, u_{(2)}, \cdots, u_{(2m)}$ are column vectors of order $2m$ and where $[u_{(i)}, u_{(j)}]=u_{(i)}u'_{(j)}-u_{(j)}u'_{(i)}$. The determinant $|U|=|u_{(1)}\cdots u_{(2m)}|$ is called the Pfaffian of $P$. The author gives simple proofs of the following known facts. (i) The determinant of every matrix $S$ which satisfies the matrix equation $S'PS=P$ is equal to unity. (ii) The Pfaffian of $P$ is uniquely determined by $P$ (inclusive of sign), and is an invariant of weight one of the bilinear form $y'Px$. An explicit expression is obtained for the Pfaffian of the matrix

$$\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

which has a bearing on problems in mechanics and optics.

*W. Ledermann* (Manchester).

**Gröbner, W. Über die Eliminationstheorie.** Monatsh. Math. 54, 71–78 (1950).

This note contains some remarks on the connection between Kronecker's elimination process and the structure of residue class rings of 0-dimensional polynomial ideals.

*O. F. G. Schilling* (Chicago, Ill.).

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## *Abstract Algebra*

**Birkhoff, Garrett. Théorie et applications des treillis.** Ann. Inst. H. Poincaré 11, 227–240 (1949).

Expository; cf. the author's Lattice Theory, revised edition, Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1948 [these Rev. 10, 673]. A few open problems are mentioned. *P. M. Whitman* (Silver Spring, Md.).

**Manises Lechuguero, Julio. New clarifications on orderable fields with a unique automorphism.** Euclides, Madrid 9, 426–427 (1949). (Spanish)

**Sikorski, Roman. On algebraic extensions of ordered fields.** Ann. Soc. Polon. Math. 22 (1949), 173–184 (1950).

The author shows that an ordinal number and a certain compactness property of an ordered field $F$ are inherited by the real closure or algebraic closure of $F$.

*I. Kaplansky* (Chicago, Ill.).

**Krishnan, Viakalathur S. L'extension d'une $(<,.)$ algèbre à une $(\sum^*,.)$ algèbre. II.** C. R. Acad. Sci. Paris 230, 1559–1561 (1950).

[For part I, see the same vol., 1447–1448 (1950); these Rev. 11, 575.] Examples of the extension mentioned in the title are given. The author also analyses the relation of the sets of elements congruent to 0 in an algebra closed under multiplication and finite distributive addition and in its extension to an algebra also closed under infinite distributive addition. *P. M. Whitman* (Silver Spring, Md.).

**Fuchs, Ladislas. A note on the descending chain condition.** Norske Vid. Selsk. Forh., Trondheim 21, no. 14, 57–59 (1949).

Un idéal $I$ d'un anneau commutatif $A$ est dit superirréductible s'il n'est pas intersection d'une famille (finie ou infinie) d'idéaux distincts de $I$. Si $A/I$ est un anneau d'Artin, et si $I$ est irréductible, alors $I$ est superirréductible. Enfin, pour que tout idéal distinct de (0) de $A$ soit intersection d'un nombre fini d'idéaux superirréductibles, il faut et il suffit que $A/I$ soit un anneau d'Artin pour tout idéal non nul $I$ de $A$; la nécessité de cette condition est démontrée seulement quand, pour les idéaux de $A$, l'opération d'addition est distributive par rapport à celle d'intersection (finie ou infinie), ce qui est une condition très restrictive.

*P. Samuel* (Clermont-Ferrand).

**Fuchs, Ladislas. On a special property of the principal components of an ideal.** Norske Vid. Selsk. Forh., Trondheim 22, no. 9, 28–30 (1949).

Soient $A$ un anneau commutatif, et $I$ un idéal de $A$. Les éléments $s\in A$ tels que, pour tout $b\in A$ non premier à $I$ (c.à.d. tel que $(I:(b))\neq I$), tout élément de $(s, b)$ soit premier à $I$, forment un idéal $S$, non premier à $I$, et contenant $I$; $S$ est appelé l'idéal adjoint de $I$. L'idéal adjoint de $I$ est l'intersection des idéaux premiers maximaux de $I$, et contient donc son radical. L'élément $b\in A$ est dit (relativement) primal à $I$ si $bc\in I$ entraîne $c\in S$; ceci généralise les notions de relativement premier et de relativement primaire. Pour que $b$ soit primal à $I$, il faut et il suffit qu'il soit extérieur aux composantes principales de $I$. *P. Samuel.*

**Brown, Bailey, and McCoy, Neal H. The maximal regular ideal of a ring.** Proc. Amer. Math. Soc. 1, 165–171 (1950).

Un élément $a$ d'un anneau $R$ est dit régulier s'il existe $x\in R$ tel que $axa=a$; un idéal bilatère est dit régulier si tous ses éléments sont réguliers. Les auteurs prouvent qu'il existe dans tout anneau $R$ un plus grand idéal régulier $M(R)$; ils montrent que $M(R/M(R))=0$ et que si $R_n$ est l'anneau des matrices carrées d'ordre $n$ sur $R$, $M(R_n)=(M(R))_n$. Désignant par $B^*$ l'annulateur (à droite et à gauche) d'un idéal bilatère $B$ de $R$, et par $J$ le radical de $R$, ils prouvent ensuite que $M\cap M^*=0$ et $J\subset M^*$; si $R/J$ est un anneau régulier, la condition $M=0$ équivaut à $J^*\subset J$. Ils montrent enfin que si $R$ satisfait à la condition minimale pour les idéaux à droite, $M$ est semi-simple, $R$ est somme directe de $M$ et $M^*$, et, dans l'anneau $M^*$, l'annulateur (à droite et à gauche) du radical est contenu dans le radical, ce qui précise un théorème de M. Hall [Trans. Amer. Math. Soc. 48, 391–404 (1940); ces Rev. 2, 122]. *J. Dieudonné.*

**Perlis, Sam, and Walker, Gordon L. Abelian group algebras of finite order.** Trans. Amer. Math. Soc. 68, 420–426 (1950).

It is determined when the group algebras of two finite Abelian groups, over a field whose characteristic does not

divide their orders, are isomorphic. If $G$ is of order $n$, the representations of the algebra $G_P$ split when extended to $G_K$, where $K$ is the root field over $F$ of $x^n-1=0$. This yields

$$(1)\qquad G\cong\sum_{d\mid n}\frac{n_d}{v_d}F(\omega_d)$$

for $\omega_d$ a primitive $d$th root of 1, $v_d$=degree $F(\omega_d)/F$, and $n_d$ equal to the number of elements of order $d$ in $G$. It follows that $G_P\cong H_P$ only if the same holds for their prime power components, thus reducing to the case that $G$ and $H$ are of the same order $p^a$. Let $p^\alpha$ be the largest invariant occurring with unequal multiplicity in $G$ and $H$, and $\zeta_1$ and $\zeta_q$ be primitive $p$- and $p^\alpha$-roots of 1. If $P$ is the prime subfield of $F$, of characteristic not $p$ (characteristic 2 requires minor changes in the statements that follow), then $P(\zeta_q)\cong P(\zeta_1)\times L$ for unique $L$. Theorem: $G_P\cong H_P$ if and only if $L\leqq F$. The proof uses (1) and a comparison of cyclotomic fields over $F$ with those over $P$. Theorem: for $F=P$, of finite characteristic, $G_P\cong H_P$ only if $q\leqq e$, where the cyclotomic $p$-number $e$ of $P$ is the least integer such that $P(\zeta_e)<P(\zeta_{e+1})$; for the rational field $R$, $G_R\cong H_R$ only if $G\cong H$. *R. C. Lyndon.*

**Ritt, J. F. Associative differential operations.** Ann. of Math. (2) **51**, 756–765 (1950).

Let $F$ be a differential field of characteristic 0, and denote the set of all differential polynomials over $F$ in $y_1, \cdots, y_m$ by $F\{y_1, \cdots, y_m\}$. The author defines a differential operation of rank $n$ as an $n$-uple $A(u, v)=(A_1(u, v), \cdots, A_n(u, v))$, where each $A_i(u, v)$ is a formal series $\sum_{j=1}^{\infty}P_{ij}$ in which $P_{ij}$ is an element of $F\{u_1, \cdots, u_n, v_1, \cdots, v_n\}$ homogeneous of degree $j$. He poses the problem of determining all differential operations of rank $n$ which are associative: $A(A(u, v), w)=A(u, A(v, w))$, $A_i(u, 0)=u_i$, $A_i(0, v)=v_i$ $(1\leqq i\leqq n)$. This problem generalizes the classification problem of Lie theory where the $A_i(u, v)$ are power series merely in $u_1, \cdots, u_n, v_1, \cdots, v_n$ and not in these and their derivatives. In the case $n=1$ an example is given by $A(u, v)=u+v$, called the operation of addition, and also by $A(u, v)=u+v+\sum_{j=1}^{\infty}v_ju^j/j!$ (where $v_j$ is the $j$th derivative of $v$). The latter is formally the expansion in powers of $u$ of the function of $x$ given by $u(x)+v(x+u(x))$ and therefore is called the operation of substitution. The author solves his problem completely for the case $n=1$; after suitably defining equivalence he proves the theorem: Every associative differential operation of rank 1 is equivalent to either the operation of addition or that of substitution. For higher ranks examples are given and development of the theory (including an analogue of Lie's third fundamental theorem) is promised for later papers. *E. R. Kolchin.*

**Barsotti, I. Errata: Algebre senza base finita.** Ann. Mat. Pura Appl. (4) **28**, 346 (1949).

Cf. the same Ann. (4) **27**, 243–260 (1948); these Rev. **10**, 676.

## THEORY OF GROUPS

**Jacobsthal, Ernst. Sur le nombre d'éléments du groupe symétrique $S_n$ dont l'ordre est un nombre premier.** Norske Vid. Selsk. Forh., Trondheim **21**, no. 12, 49–51 (1949).

Let $A_{n,p}$ be the number of permutations of degree $n$ and order $p$. The author proves that

$$A_{n,p}=n!\sum_{\lambda=1}^{[n/p]}\{(n-\lambda p)!\lambda!p^\lambda\}^{-1}.$$

Putting $w_{n,p}=A_{n,p}/n!$ he shows that

$$e^{x(e^{xp}/p-1)}=\sum_{n=1}^{\infty}w_{n,p}x^n,$$

whence an asymptotic expression for $w_{n,p}$ may be deduced. *W. Ledermann* (Manchester).

**Zassenhaus, Hans. An equation for the degrees of the absolutely irreducible representations of a group of finite order.** Canadian J. Math. **2**, 166–167 (1950).

Let $G$ be an arbitrary finite group of order $N$, $S$ the group-algebra of $G$ over the rational number field, $r$ the number of classes of conjugate elements of $G$, $C_i$ the sum and $h_i$ the number of the elements of the $i$th class, and finally $f^i$ the degree of the $i$th absolutely irreducible representation of $G$ $(i=1, \cdots, r)$. Denote the element $N\sum_{i=1}^{r}\sum_{k=1}^{r}(\delta_{ik'}/h_i)C_iC_k$ of $S$ by $C$, the $k'$th class being the inverse of the $k$th class. In the regular representation of $S$, to the element $C$ an integral matrix $R(C)$ is attached, the elements of which can be computed from the multiplication table of $G$. The author shows that the characteristic polynomial $f(t)$ of the matrix $R(C)$ is equal to $\prod_{i=1}^{r}(t-(N/f^i)^2)^{(f^i)^2}$, which means that the numbers $N/f^i$ can be obtained by solving an equation explicitly known. Since the coefficients of that equation are rational integers and the largest coefficient is 1, it follows that $f^i$ is a divisor of $N$. *T. Szele* (Debrecen).

**Gel'fand, I. M., and Cetlin, M. L. Finite-dimensional representations of groups of orthogonal matrices.** Doklady Akad. Nauk SSSR (N.S.) **71**, 1017–1020 (1950). (Russian)

The authors give explicit formulas for the irreducible finite-dimensional representations of the Lie algebra of all skew-symmetric matrices of a given finite order, apparently in continuation of their similar work on the unimodular group. As the foregoing Lie algebra is that of the orthogonal group, the irreducible finite-dimensional representations of these groups are thereby determined. Actually, relatively explicit formulas for these representations are classical.

*I. E. Segal* (Chicago, Ill.).

**Gotô, Ken-iti. Spinors in five dimensions.** Progress Theoret. Physics **5**, 42–55 (1950).

The four-component spinor representation of the five-dimensional orthogonal group is studied in terms of the transformations induced on a pair of two component spinors. The wave equations satisfied by a second order four-component spinor are discussed.

*A. H. Taub* (Urbana, Ill.).

**Schouten, J. A. On the geometry of spin spaces. IV.** Nederl. Akad. Wetensch., Proc. **53**, 261–272 = Indagationes Math. **12**, 41–52 (1950).

[For the first three parts see same Proc. **52**, 597–609, 687–695, 938–948 = Indagationes Math. **11**, 178–190, 217–225, 336–346 (1949); these Rev. **11**, 157.] This part of the author's review of spinor theory is mainly concerned with the characterisation of the various groups in the spin spaces which correspond to the subgroups of the orthogonal group in $n=2\nu$ or $n=2\nu-1$ dimensions; for example, the real orthogonal group, the proper orthogonal group.

*A. H. Taub* (Urbana, Ill.).

**Kuranishi, Masatake. Two elements generations on semi-simple Lie groups.** Kōdai Math. Sem. Rep., no. 5–6, 9–10 (1949).

If $G$ is a connected semi-simple Lie group then $G$ contains two elements which generate an everywhere dense subgroup of $G$. The proof makes use of results on Lie algebras and the following lemma: If $L$ is a local Lie group which can be approximated by discrete subgroups, then $L$ is nilpotent.

*D. Montgomery* (Princeton, N. J.).

**Borel, Armand. Limites projectives de groupes de Lie.** C. R. Acad. Sci. Paris **230**, 1127–1128 (1950).

This note begins with some general remarks about projective limits which often make unnecessary the assumption of denumerability, and then makes applications to the case where a topological group $G$ is the projective limit of Lie groups $G_\alpha$. If $g_\alpha(t)$ is a one-parameter group in $G_\alpha$, then $G$ contains a one-parameter group which is mapped on $g_\alpha(t)$. If $G$ is connected, the subset connected to the identity by an arc is a subgroup everywhere dense in $G$. If $S_\alpha$ and $S_\alpha^1$ are two coherent systems of maximal semisimple subgroups their limits are conjugate. If $G$ is locally compact there is an index $\nu$ such that the kernel of $f_{\alpha\beta}$, $\nu<\alpha<\beta$, is compact.

*D. Montgomery* (Princeton, N. J.).

**Borel, Armand. Le plan projectif des octaves et les sphères comme espaces homogènes.** C. R. Acad. Sci. Paris **230**, 1378–1380 (1950).

Let $A_i$, $B_i$, $C_i$, $D_i$ denote the four systems of classical compact Lie groups and let $\bar{B}_i$ and $\bar{D}_i$ be the simply connected covering groups of $B_i$ and $D_i$. Let $G_2$ and $F_4$ be the usual exceptional simple groups. The projective plane of Cayley numbers is homeomorphic to the homogeneous space $F_4/\bar{B}_4$. The space $G_2/D_2$ is not homeomorphic to the projective plane of quaternions but the two have the same Poincaré polynomial. Let $G$ be a compact connected Lie group which acts effectively on $W=G/G^1$, $G^1$ connected. If rank $G^1$ = rank $G-k$ and if $G$ is locally isomorphic to the direct product of at most $k+1$ nontrivial groups, then $W$ is a direct product of homogeneous spaces. The homogeneous spaces of simple compact Lie groups which are homeomorphic to $S_{2n-1}$ are $A_n/A_{n-1}$; $D_n/B_{n-1}$; for $n=2m$, $C_m/C_{m-1}$; $\bar{B}_4/\bar{B}_3=S_{15}$ and $\bar{B}_3/G_2=S_7$. There are also results on homotopy of groups.

*D. Montgomery* (Princeton, N. J.).

**Følner, Erling. A proof of the simpler Pontrjagin duality theorems by help of the connection between two infinite-dimensional spaces.** Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 19, 15 pp. (1950).

The author presents a proof of the Pontrjagin duality theorems for locally compact Abelian groups in the special case in which the dual groups are respectively discrete and compact and both are separable. He first uses the fact that a compact Abelian group has sufficiently many characters to imbed it in an infinite Cartesian product of circle groups. Then he takes advantage of the fact that the latter group is a homomorphic image of a certain topological vector space and applies a vector space duality theorem worked out earlier in collaboration with H. Bohr [Courant Anniversary Volume, pp. 45–62, Interscience Publishers, New York, 1948; these Rev. **10**, 507]. *G. W. Mackey.*

**Birkhoff, Garrett. Groupes réticulés.** Ann. Inst. H. Poincaré **11**, 241–250 (1949).

Brief exposition of the theory and some applications of lattice-ordered groups. *P. M. Whitman.*

**Jaffard, Paul. Nouvelles applications de la théorie des filets.** C. R. Acad. Sci. Paris **230**, 1631–1632 (1950).

[Cf. the author's papers, same C. R. **230**, 1024–1025, 1125–1126 (1950); these Rev. **11**, 579.] Let $G$ denote a lattice ordered group, and $F$ the set of its filets. If $F$ satisfies the weak ascending chain condition then $F$ is relatively complemented. The converse holds if the descending chain condition also holds. $G$ is the direct sum of linearly ordered groups if and only if $F$ is relatively complemented and satisfies the descending chain condition and, given filets $a^*$, $b^*\neq 0$, there exists $x$ in $a^*$ such that $x\not\equiv b$. There are additional results on uniqueness and irreducibility of representations as a subgroup of a product of linearly ordered groups. *P. M. Whitman* (Silver Spring, Md.).

**Lorenzen, Paul. Über halbgeordnete Gruppen.** Arch. Math. **2**, 66–70 (1949).

The author sketches a lattice-theoretic version of multiplicative ideal theory, much of which can be generalized to noncommutative groups. A full treatment will appear later. The results announced include: Every principal ideal ring can be expressed as an intersection of valuation rings; a semiordering of a semiordered group $G$ can be represented as a join of orders of $G$, if and only if $G$ can be extended to a regular 1-group; every closed subsemigroup is an intersection of prime subsemigroups. *G. Birkhoff.*

## NUMBER THEORY

* **Nagell, Trygve. Elementär talteori. [Elementary Number Theory].** Almqvist & Wiksells Akademiska Handböcker, Hugo Gobers Förlag, Uppsala, 1950. 271 pp. 21 kronor.

A lucidly written short introduction into the fundamentals of number theory for students without any preparation. The author leads up to quadratic number fields and roots of unity, but these notions are used in a most elementary way. Occasionally vistas are opened such as a description of the prime number problem or the use of the Thue-Siegel theorem for the problem of lattice points on algebraic curves. Many examples and problems are given. The chapter headings are: Divisibility properties [36 pp.]; On the distribution of primes [18 pp.]; Theory of congruences [57 pp.]; Theory of quadratic residues [21 pp.]; Arithmetic properties of roots of unity [21 pp.]; Diophantine equations of second order [30 pp.]; Diophantine equations of higher order [33 pp.]; Additional problems, Tables of the number of primitive roots of the first 150 primes and of the fundamental solution of $x^2-Dy=\pm 1$; Chronological list of authors.

*W. Feller* (Princeton, N. J.).

**Nagell, Trygve. Sur les restes et les non-restes quadratiques suivant un module premier.** Ark. Mat. **1**, 185–193 (1950).

Let $\pi_p$ denote the smallest odd prime which is a quadratic residue of $p$, and similarly $\psi_p$ the smallest odd prime quadratic nonresidue. The paper continues an earlier discussion

by the author [Skrifter utgit av Videnskapsselskapet i Kristiania. I. Math.-Nat. Kl. 1923, no. 13 (1924)] in which it was established that if $p=4n+1$ then $\pi_p<p^{\frac{1}{2}}$ for $p>17$. It is now proved that if $p=8n+1$, then $\psi_p<p^{\frac{1}{2}}$; if $p=8n+5$, $\psi_p<(2p)^{\frac{1}{2}}$; if $7<p=8n-1$, $\pi_p\leqq 2p^{\frac{1}{2}}-1$ and $\psi_p<(2p)^{\frac{1}{2}}-1$. Moreover, there is a prime quadratic nonresidue $q\equiv 3 \pmod 4$ belonging to $p$ such that $q<2p^{\frac{1}{2}}-1$ if $p=8n-1$ and $q<2p^{\frac{1}{2}}+1$ if $p=8n+3$. Finally for $p=8n+3$ we have $\pi_p\leqq\{(p+16)/3\}^{\frac{1}{2}}-2$ except for $p=3$, 11, 19, 43, 67, 163 and perhaps one other value of $p$. *I. Niven.*

**Sierpiński, W. Contribution à l'étude des restes cubiques.** Ann. Soc. Polon. Math. **22** (1949), 269–272 (1950).

Theorem. Let $q$ be a prime number and $m>1$, a natural number. In order that there exist for every integer $x$ an integer $y$ such that $x\equiv y^q \pmod m$, it is necessary and sufficient that $m$ be a product of distinct primes none of which is of the form $qk+1$. Sketch of proof. From elementary group-theoretic considerations it is easy to see that every integer prime to $m$ is a $q$th-power residue if and only if $m$ is not divisible by either $q^2$ or any prime of the form $qk+1$ [cf. Meyer, Arch. Math. Phys. **43**, 413–436 (1865)]. For $m$ not divisible by $q^2$ or by any prime of the form $qk+1$, the residue classes not prime to $m$ give trouble only if $m$ is nonsquarefree. *P. T. Bateman* (Urbana, Ill.).

**Jacobsthal, Ernst. Beiträge zur Zahlentheorie. I.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 30, 137–142 (1950).

Let $r_b$ be the least residue of $km$ modulo $n$, where $(m, n)=1$. Denote by $D(a, x, y)$ the number of elements among $r_1, r_2, \cdots, r_a$ which lie in the closed interval $(x, y)$. Denote $r_a$ by $b$. If $b<n/2$, then $D(2a, 2b, n-1)=2D(a, b-1, n-b-1)$; if $b\geqq n/2$, then

$$2a+1-D(2a, 2b-n+1, n-1)=2D(a, n-b, b).$$

If $n>m$, $n$ and $m$ odd, $a=(n-1)/2$, then

$$D(a, (n-m)/2, (n+m)/2)=\tfrac{1}{2}(m+1).$$

If $n>m$, $n$ odd, $m$ even, $a=\frac{1}{2}(n-1)$, then

$$D(a, m/2, (2n-m)/2)=\tfrac{1}{2}(n+1-m).$$

If $n>2m$, $n$ even, $m$ odd, $a=\frac{1}{2}n-1$, then

$$D(a, \tfrac{1}{2}n-m, \tfrac{1}{2}n+m)=m.$$

If $m<n<2m$, $n$ even, $m$ odd, $a=\frac{1}{2}n-1$, then

$$D(a, m-\tfrac{1}{2}n, \tfrac{3}{2}n-m)=n-m.$$

*I. Niven* (Eugene, Ore.).

**Jacobsthal, Ernst. Beiträge zur Zahlentheorie. II.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 31, 142–148 (1950).

This continues the study of $D(a, x, y)$ [see the preceding review for definitions]. A general formula is developed for $D$ in terms of $n$, $a$, $b$, $x_1$, and $t$, where $x_1$ is defined by $r_{x_1}=n-x$, and $t$ is the least positive integer such that $tm\equiv x-y \pmod n$. *I. Niven* (Eugene, Ore.).

**Jacobsthal, Ernst. Beiträge zur Zahlentheorie. III.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 32, 149–153 (1950).

[See the preceding two reviews for notation.] Let $\|x\|=-1$, 0, or 1 according as $x<0$, $0\leqq x<1$, or $x\geqq 1$. If $m+1\leqq x\leqq n-1$, then

$$D(a, x-m, x)=[am/n]+[(x_1+a)/n]+\|(r_a+m)/x\|.$$

Similar formulas are developed in the following cases: for $D(a, x+m-n, x)$ in case

$$n-1\geqq x\geqq n-m+1;$$

for $D(a, x-2m, x)$ in case

$$2m+1\leqq x\leqq n-1;$$

for $D(a, x+n-2m, x)$ in case $n<2m$ and

$$2m-n+1\leqq x\leqq n-1;$$

for $D(a, x+2m-n, x)$ in case $n>2m$ and

$$n-2m+1\leqq x\leqq n-1;$$

for $D(a, x+2m-2n, x)$ in case $n<2m-1$ and

$$2n-2m+1\leqq x\leqq n-1.$$

*I. Niven* (Eugene, Ore.).

**Jacobsthal, Ernst. Beiträge zur Zahlentheorie. IV.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 33, 154–158 (1950).

The results of this paper stem from the same methods as in the previous papers [see the three preceding reviews]. Let $f(m, n)=\sum[km/n]$, where the sum ranges over $k=0, 1, \cdots, [\frac{1}{2}(n-1)]$. With $(m, n)=1$, and $1<m<n$, it is proved that $f(m, n)<f(n, m)$ if $n$ is even; that $f(m, n)\geqq f(n, m)$ if $m$ is even; that $f(m, n)\leqq f(n, m)$ if both $m$ and $n$ are odd. *I. Niven* (Eugene, Ore.).

**Rédei, L. Elementarer Beweis und Verallgemeinerung einer Reziprozitätsformel von Dedekind.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 236–239 (1950).

If $F_n(x)=x^{n-1}+\cdots+x+1$ and if the positive integers $m$ and $n$ are relatively prime, there are uniquely determined polynomials $X_{mn}(x)$ and $X_{nm}(x)$ of degrees $n-1$ and $m-1$ respectively such that: (A) $F_n(x)X_{mn}(x)+F_m(x)X_{nm}(x)=1$. They are given by:

$$X_{mn}=\sum_{k=0}^{n-1}(-\{km'/n\}+\{(k-1)m'/n\}+\{-m'/n\}-\{-2m'/n\})x^k,$$

if the integers $m'$ and $n'$ are chosen such that $mm'+nn'=1$ and where $\{z\}=z-[z]-\frac{1}{2}$ ($[z]$ is the largest integer $\leqq z$). Now, taking $x=1+t$ in (A) and equating the coefficients of $t^2$, the reciprocity formula

$$S_{mn}+S_{nm}=(12mn)^{-1}(m^2-3mn+n^2+1)$$

can be obtained, where $S_{mn}=\sum_{k=1}^{n-1}\{k/n\}\{mk/n\}$.

*H. D. Kloosterman* (Leiden).

**Apostol, T. M. Generalized Dedekind sums and transformation formulae of certain Lambert series.** Duke Math. J. **17**, 147–157 (1950).

In the theory of the transformation of Dedekind's modular function $\log\eta(\tau)$ there appears a certain arithmetic sum,

$$s(h, k)=\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right),$$

where $h$, $k$ with $(h, k)=1$ are coefficients of the modular substitution and $[x]$ is the largest integer not exceeding $x$. A reciprocity law $12s(h, k)+12s(k, h)=-3+h/k+k/h+1/hk$ was proved by Dedekind using function-theoretic methods. More recently these sums were treated arithmetically in a series of papers by Rademacher [cf. especially Rademacher and Whiteman, Amer. J. Math. **63**, 377–407 (1941); these Rev. **2**, 249]. The author generalizes these sums to $s_p(h, k)=\sum_{\mu=1}^{k-1}\mu k^{-1}\overline{B}_p(h\mu/k)$, $p=$integer $\geqq 1$, where $\overline{B}_p$ is the Bernoulli polynomial $\overline{B}_p(x)=\sum_{s=0}^{p}B_s(p_s)(x-[x])^{p-s}$, $B_s$ being the Bernoulli numbers defined recursively by $B_0=1$, $\sum_{s=0}^{p}\binom{p}{s}B_s=B_p$. (Note that $\overline{B}_1(x)=x-[x]-\frac{1}{2}$.) A general-

ized reciprocity law is proved:

$$(p+1)\{hk^p s_p(h,k)+kh^p s_p(k,h)\}$$
$$=pB_{p+1}+\sum_{s=0}^{p+1}\binom{p+1}{s}(-1)^s B_s h^s B_{p+1-s}k^{p+1-s},$$

$p$ odd. Just as $s(h,k)$ appears in the transformation of

$$\log\eta(\tau)=\frac{\pi i\tau}{12}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{-1}x^{mn}=\frac{\pi i\tau}{12}-G_1(x),$$

$x=e^{2\pi i\tau}$, so $s_p(h,k)$ is connected with the functions $G_p(x)$ for which the author develops a transformation formula using Rademacher's method [J. Reine Angew. Math. 167, 312–336 (1932)]. Furthermore, he expresses $s_p(h,k)$ as a sort of Lambert series, generalizing a result of Dedekind for $p=1$, and thus arrives at a closed expression for $s_p(h,k)=\frac{1}{2}(k-k^p)k^{-p}B_p$ when $p$ is an even integer.

*J. Lehner* (Philadelphia, Pa.).

**Rademacher, H. Die Reziprozitätsformel für Dedekindsche Summen.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 57–60 (1950).

Calculating the Stieltjes integral $\int_{\epsilon}^{1-\epsilon}((x))d((hx))((kx))$, $\epsilon>0$, in two different ways, the author obtains a simple proof for the reciprocity formula

$$s(h,k)+s(k,h)=(12hk)^{-1}(h^2-3hk+k^2+1)$$

for the Dedekind sums $s(h,k)=\sum_{\mu=1}^{k-1}((\mu/k))((h\mu/k))$, where $((x))=x-[x]-\frac{1}{2}$ or 0 according as $x$ is not or is an integer.

*H. D. Kloosterman* (Leiden).

**Steuerwald, Rudolf. Über die Kongruenz $a^{n-1}\equiv 1 \pmod n$.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 69–70 (1949).

A short proof is given of the known fact that for each $a$ there are infinitely many exceptions to the strict converse of Fermat's theorem: if $m$ divides $a^m-a$, then $m$ is a prime.

*D. H. Lehmer* (Berkeley, Calif.).

**Ramanathan, K. G. Identities and congruences of the Ramanujan type.** Canadian J. Math. **2**, 168–178 (1950).

Let $\{(1-x)(1-x^2)\cdots\}^{-\nu}=\sum_{n=0}^{\infty}P_\nu(n)x^n$, and

$$G_{\nu,p}(x)=\sum_{l=0}^{\infty}P_\nu(p^\alpha l+\rho)x^l,\quad p=5,7;\quad 24\rho\equiv\nu \pmod{p^\alpha}$$

with $\rho$ minimal positive. (When $\nu=1$, $P_\nu(n)$ reduces to $p(n)$, the number of unrestricted partitions of $n$.) The author, using methods and results of Watson [J. Reine Angew. Math. 179, 97–128 (1938)] and Rademacher [Trans. Amer. Math. Soc. 51, 609–636 (1942); these Rev. 3, 271], establishes identities for $G_{\nu,p}(x)$ and congruences for $P_\nu(n)$ analogous to and including those of Ramanujan and Watson. Examples: (1) if $24m\equiv\nu \pmod{5^\alpha}$ with $\nu\equiv 16, 21$, or $26 \pmod{30}$, then $P_\nu(m)\equiv 0 \pmod{5^u}$ where $u=[(\alpha+1)/2]$;

(2) $$G_{2,5}(x)\equiv\sum_{l=0}^{\infty}P_3(5l+2)x^l=x^{-1}\prod_{1}^{\infty}(1-x^{5m})^{-5}$$
$$\times\{9\varphi+75.5\varphi^3+125.5^3\varphi^5\},$$

where

$$\varphi=x\prod_{1}^{\infty}(1-x^{5m})^6(1-x^m)^{-6}.$$

He remarks that results for $p=13$ generalizing those of Zuckerman [Duke Math. J. 5, 88–110 (1939)] can also be obtained by his method.

*J. Lehner* (Philadelphia, Pa.).

**Erdös, P. On a problem in elementary number theory.** Math. Student **17** (1949), 32–33 (1950).

Let $S(n)=\sum\mu(d)$, where the summation is over $d|n$, $a\leqq d\leqq b$, and $\mu(d)$ is the Möbius function. It is proved that

$$S(n)\leqq\binom{v(n)}{[v(n)/2]}$$

for arbitrary positive values $a$ and $b$, $v(n)$ being the number of prime divisors of $n$. For a given positive integer $k$ it is shown that the equality sign holds for infinitely many $n$, with $v(n)=k$.

*I. Niven* (Eugene, Ore.).

**Ryde, Folke. Les quantités irrationnelles quadratiques et les substitutions linéaires.** Ark. Mat. **1**, 179–184 (1950).

Let $(\alpha\theta+\beta)/(\gamma\theta+\delta)$ be a linear substitution in which the integers $\alpha$, $\beta$, $\gamma$, $\delta$ are subject only to the restriction $\alpha\delta-\beta\gamma\neq 0$, and $\theta$ satisfies the irreducible quadratic equation $p\theta^2+q\theta+r=0$ where the integers $p$, $q$, $r$ have no common factor. The author shows that it is possible to determine, in a double infinity of ways, another substitution $(A\theta+B)/(C\theta+D)$ where $A$, $B$, $C$, $D$ are integers, and where the quadratic equation above has the form

$$(A\theta+B)(\gamma\theta+\delta)-(C\theta+D)(\alpha\theta+\beta)=0.$$

The method involves the solution of a system of linear Diophantine equations.

*W. H. Gage* (Vancouver, B. C.).

**Patz, Wilhelm. Über die Gleichung $X^3-DY^3=\pm c\cdot(2^{31}-1)$, wo $c$ möglichst klein.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 21–30 (1949).

The author illustrates by three examples a continued fraction method of solving $x^2-Dy^2=cp$. The examples considered are $x^2+3y^2=p$, $x^2+5y^2=2p$, and $x^2-13y^2=-p$, where $p=2^{31}-1$. Fermat's theorem is used to solve $z^2\equiv D \pmod p$ and then the convergents of the continued fraction for $(D^{\frac{1}{2}}-z)/p$ give the solution $(x,y)$.

*D. H. Lehmer.*

**Moessner, Alfred. Some Diophantine problems.** Euclides, Madrid **9**, 423–426 (1949). (Spanish)

Various relations among sums of powers of integers are obtained which are based essentially upon a parametric solution of the Diophantine system $3A=B+C+D$, $A^3=BCD$.

*H. W. Brinkmann* (Swarthmore, Pa.).

**Skolem, Th. On the Diophantine equation $ax^2+by^2+cz^2+du^2=0$.** Norske Vid. Selsk. Forh., Trondheim **21**, no. 19, 76–79 (1949).

A new proof is given of the theorem that

$$f(x,y,z,u)=ax^2+by^2+cz^2+du^2=0$$

is solvable under the hypotheses that $a$, $b$, $c$, $d$ are square-free and not all of the same sign, that $abcd$ contains no cube and not the factor 4, that $f\equiv 0 \pmod 8$ is solvable, and that $-ab$, $-ac$, $-ad$, $-bc$, $-bd$, $-cd$ are quadratic residues of $(c,d)$, $(b,d)$, $(b,c)$, $(a,d)$, $(a,c)$, $(a,b)$ respectively. The proof is a simplification of an earlier proof by the author [Diophantische Gleichungen, Springer, Berlin, 1938]. For example, no use is now made of Dirichlet's theorem on the existence of primes in an arithmetic progression.

*I. Niven* (Eugene, Ore.).

**Palamà, Giuseppe. Somme uguali di biquadrati.** Boll. Un. Mat. Ital. (3) **4**, 417–422 (1949).

A solution in integers is obtained for the equation

$$x_1^4+\cdots+x_n^4=y_1^4+\cdots+y_m^4$$

when $n\geqq 2$, $m\geqq 3$.

*W. H. Simons* (Vancouver, B. C.).

**Palamà, G. Multigrade con termini uguali o primi. Generalizzazione di teoremi delle multigrade.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 60–76 (1949).

A multigrade equation of the type

$$m_1A_1^k+\cdots+m_\rho A_\rho^k=q_1\beta_1^k+\cdots+q_\sigma\beta_\sigma^k, \qquad k=n_1, \cdots, n_\tau$$

is denoted by

$$[m_1]A_1, \cdots, [m_\rho]A_\rho \overset{k}{=} [q_1]\beta_1+\cdots+[q_\sigma]\beta_\sigma, \quad k=n_1, \cdots, n_\tau.$$

In the first part of the paper equations of the type

$$A_1, \cdots, [m]A_n \overset{4}{=} \beta_1, \cdots, [m]\beta_n$$

are obtained by applying Tarry's theorem to identities of the form $m_1^2+\cdots+m_n^2=p^2$, the $M_i$, $A_i$, $B_i$ being quadratic forms. In part II are given numerical examples of multigrade equations whose terms are all primes. Finally, in part III, compound multigrade equations are defined and the theorems of Tarry and Frolov are generalized for such equations. [The paper contains some typographical omissions. The last term of equation (5) page 62 should be $\{3, 2, -1\}$. The fourth line from the bottom of page 62 should be $a_{i+1}=a_i+\alpha$. In the fifth line from the bottom of page 63 $r$ should be $\{1, 0, -11\}$.] *W. H. Simons.*

**Cassels, J. W. S. The Markoff chain.** Ann. of Math. (2) **50**, 676–685 (1949).

This paper gives a simple proof of the theorem by A. Markoff [Math. Ann. **15**, 381–406 (1879); **17**, 379–399 (1880)] on the successive minima of indefinite binary quadratic forms. The theorem is formulated as follows. Let $f(x, y)=ax^2+bxy+cy^2$ be a real binary form of positive discriminant $d=b^2-4ac$, and let 1 be the lower bound of $|f(x, y)|$ for integers $x, y$ not 0, 0. Then if $d<9$, $f$ is equivalent to a form $\phi_P=(x+hy)^2-ky^2$ ($h=3/2-Q/P$, $k=9/4-1/P^2$), where $P$ is a solution of the Diophantine equation (i) $P^2+P_1^2+P_2^2=3PP_1P_2$ ($P\geqq P_1$, $P\geqq P_2$), and $Q$ is the least nonnegative solution of $\pm P_1Q\equiv P_2 \pmod P$. The proof is a considerable simplification of Remak's [Math. Ann. **92**, 155–182 (1924)], avoiding continued fractions. The first three minima are obtained by a method modeled on one by Oppenheim [Quart. J. Math. Oxford Ser. (1) **3**, 10–14 (1932)]. Properties of the solutions of (i) are developed in part 2, and these are applied to complete the proof by induction, in part 3. *G. Pall* (Chicago, Ill.).

**Jones, B. W. The composition of quadratic binary forms.** Amer. Math. Monthly **56**, 380–391 (1949).

Let $\Delta$ be a nonsquare integer congruent to 0 or 1 (mod 4), and write $\sigma=\frac{1}{2}(\epsilon+\Delta^t)$, where $\epsilon=0$ or 1 according as $\Delta$ is even or odd. The ring $J(\Delta)$ of numbers $a+b\sigma$ (with $a$ and $b$ any rational integers) is studied. In previous treatments [for example; Landau, Vorlesungen über Zahlentheorie, v. III, Hirzel, Leipzig, 1927, pp. 187–196], $\Delta$ was assumed to be divisible by no square $s^2$ such that $\Delta/s^2$ is again a discriminant, and a correspondence was set up between classes of ideals in $J(\Delta)$ and classes of nonnegative primitive binary quadratic forms of discriminant $\Delta$. In the more general case of this article there is similarly set up a correspondence between classes of ideals in $J(\Delta)$ and classes (not necessarily primitive) of binary quadratic forms of discriminant $\Delta$. Composition of primitive classes is shown to map into the multiplication of the corresponding classes of ideals. Some consequences relating to composition of binary quadratic forms are deduced. *G. Pall* (Chicago, Ill.).

**Jones, Burton W. Representations by quadratic forms.** Ann. of Math. (2) **50**, 884–899 (1949).

Capital letters will denote integral matrices, $A$ and $B$, symmetric of orders $n$ and $m$, and determinants $d$ and $q$. If $T_1$ is a solution of $T_1^TAT_1=B$ (i.e., a representation of $B$ by $A$), the representations $PT_1$ (where $P$ ranges over the unimodular automorphs of $A$) form a set of "essentially equal" representations. If $T_1$ is primitive, $T_2$ can be chosen so that $T=(T_1\ T_2)$ is unimodular. The matrix $Q=T^TAT$ has $B$ as leading minor. Completing squares with reference to $B$ yields the matrix $E=qD-C^T(\text{adj}\ B)C$ (where $C$ is the minor of $Q$ to the right of $B$). The class of $C$ depends only on $T_1$, not on $T_2$. A certain set of possible matrices $C$ is defined. With each of these, and with a set of "essentially equal" solutions of the congruence $X^T(\text{adj}\ B)X\equiv -E \pmod q$, is associated, conversely, a set of primitive representations by a certain class of forms. The possible classes may be those of the same determinant as $A$, or if $q$ is assumed prime to $2d$, those of the genus of $A$. The resulting equality is shown to be particularly simple if $m=n-1$, or if $m=1$ and $q$ is prime to $2d$. The matrices $E$ which can arise for a given genus of matrices $A$ is explicitly described in the case $n=3$. This leads to the following particular case of the general theory. Let $h(\Delta)$ denote the number of classes of properly primitive binary quadratic forms of determinant $\Delta$; let $t$ denote the greatest common divisor of the second-order minor determinants of $A$, $\Delta=qd/t^2$, $q$ prime to $2d$, $q\neq\pm 1$; let $\mu(w)$ be the number of distinct odd prime factors of $w$. Then the number of sets of primitive representations of the number $q$ by the genus of the ternary matrix $A$ is $2^{-\mu(d/t^2)}h(\Delta)\rho$ or 0, where $\rho=\frac{1}{2}$ if $\Delta\equiv 1$ or 2 (mod 4) or 4 (mod 8), $\rho=2$ if $\Delta\equiv 7 \pmod 8$ and $t$ is odd, $\rho=1$ if $\Delta\equiv 7 \pmod 8$ and $t$ is even; if $\Delta\equiv 3 \pmod 8$, $\rho=\frac{3}{4}$ unless one of the three cases ((i) $c_2(A)=(-1)^r$, where $2^r$ is the highest power of 2 in $t$, (ii) $\Delta=3$, (iii) $c_3(A)\neq(-1)^r$, $\Delta<0$, and $vu$ is odd in the fundamental solution of $v^2+\Delta u^2=4$) holds, when $\rho=1$; $\rho=\frac{1}{4}$ if $\Delta\equiv 0 \pmod 8$; if $\Delta<0$ and $v\not\equiv\pm 1 \pmod q$ in the fundamental solution of $v^2+\Delta u^2=1$, the preceding value of $\rho$ is to be divided by 2. The number of sets of representations of an odd number by $xy+xz+yz$ is considered as a further application. There is a close affinity between this article and one by the reviewer [see the following review]. *G. Pall* (Chicago, Ill.).

**Pall, Gordon. Representation by quadratic forms.** Canadian J. Math. **1**, 344–364 (1949).

Let $A$ and $B_1$ be symmetric matrices of orders $n$ and $k$, respectively, having real elements, $1\leqq k\leqq n$; $T_1$ is a matrix with integer elements for which $T_1AT_1^T=B_1$, $T_1$ being primitive, i.e., 1 being the g.c.d. of its $k$-rowed minors. There is a matrix $T_2$ with integer elements such that $T=(T_1\ T_2)$ is unimodular, that is, has determinant 1. Then

$$B=T^TAT=\begin{bmatrix} B_1 & K^T \\ K & B_2 \end{bmatrix}, \quad G+K(\text{adj}\ B_1)K^T=B_1B_2$$

define matrices $B$, $K$, $B_2$, and $G$. A fundamental theorem is theorem 3. Let $A$, $T_1$, $B_1$, $G$, and $K$ be defined as above. Let $\Gamma_1(A, T_1)$ denote the subgroup of unimodular automorphs $W$ of $A$ such that $WT_1=T_1$, and $\Gamma_2(G, K)$ the subgroup of unimodular automorphs $U$ of $G$ such that $U^TK$ and $K$ are congruent modulo $B_1$. The two subgroups are isomorphic.

The author derives various consequences of this result including the following which fills a gap in a proof of Linnik on the representations of large numbers by ternary quad-

ratic forms. Consider a representative set of forms $f_1, \cdots, f_s$ with integral matrices of a given nonzero determinant $d$, and an integral binary quadratic form $\varphi = k\varphi_1$ ($\varphi_1$ properly or improperly primitive) of determinant $b_1 = k^2\Delta$, where $k$ is prime to $d$. Let $h^2$ denote the largest square factor common to $k$ and $\Delta$. Let $\rho$ denote the number of sets of representations of $\varphi$ by $f_1, \cdots, f_s$ such that the divisor of the representations is $k$. Then, for any positive $\epsilon$, there exists a constant $q$, depending on $\epsilon$ and $d$, but independent of $b_1$ and $\varphi$, such that $\rho < qh(b_1)^\epsilon$.

Theorem 3 is very similar to a result of the reviewer [see the preceding review] appearing almost simultaneously and independently derived; but the applications are very different. *B. W. Jones* (Boulder, Colo.).

**Rogers, C. A. A problem of Hirsch.** Bull. Soc. Roy. Sci. Liège 17, 263–280 (1948).

G. Hirsch conjectured that under certain hypotheses, if $(a^y - p)/(b^y - q)$ is an integer for all positive integers $y$, then there is an integer $d$ ($\geqq 0$) such that $a = b^d$ and $p = q^d$. This is proved here under the (less restrictive) hypotheses that $a$ and $b$ are integers, $a \neq 0$, $|b| > 1$, $p$ and $q$ ($\neq 0$) real, with $y$ restricted to $N \leqq y \leqq N + h + 1$, where $h$ is a nonnegative integer such that $|b|^{h+1} > |a|$, and $N$ is sufficiently large; the conclusion is that $a = b^d$, $p = q^d$ for some $d$ ($0 \leqq d \leqq h$) and that $q$ is an integer if $d > 1$. The following theorems of a more general type are proved. (I) Let $P(x)$, $Q(x)$ be polynomials of degrees $n$ and $m$ with real coefficients. The function $P(x)/Q(x)$ is a polynomial (and is integral-valued for all integers $x$) if $P(x)/Q(x)$ is integral for $x = N, N+1, \cdots, N+n$, for $N$ sufficiently large. (II) Assume $P(x)$ not identically zero, $Q(x)$ not a monomial, $a$ and $b$ integers numerically exceeding 1, $h$ a nonnegative integer such that $|b|^{m+h} > |a|^n$; write $H = (n+1)(m+h)$. If $P(a^y)/Q(b^y)$ is integral for $y = N, N+1, \cdots, N+H$ ($N$ being even and sufficiently large), there exist positive integers $c, d$ ($< m+h$), and $e$ ($\leqq n$) such that $a = \pm c^d$ and $b = \pm c^e$; also, $P(a^y)/Q(b^y)$ is integral for $y \geqq N$, and $P(x^d)/Q(x^e)$ and $P(ax^d)/Q(bx^e)$ are polynomials in $x$ having a certain (explicitly stated) form. The proofs are elementary.

*G. Pall* (Chicago, Ill.).

**Kuipers, L. Prime-representing functions.** Nederl. Akad. Wetensch., Proc. 53, 309–310 = Indagationes Math. 12, 57–58 (1950).

It is proved that for each positive integer $c \geqq 3$ there exists a real number $A = A(c)$ such that $[A^{c^x}]$ is a prime for every positive integer $x$. This generalizes a result of Mills [Bull. Amer. Math. Soc. 53, 604 (1947); these Rev. 8, 567], who treated the case $c = 3$. *I. Niven.*

**Linnik, Yu. V. A remark on products of three primes.** Doklady Akad. Nauk SSSR (N.S.) 72, 9–10 (1950). (Russian)

The author sketches a proof of the following result. If $x$ is large, then in the interval $[x, x + x^{\frac{1}{2}} \exp\{(\log x)^{0.99}\}]$ there exist integers which are products of exactly three prime factors. *P. T. Bateman* (Urbana, Ill.).

**Erdös, P., and Rényi, A. Some problems and results on consecutive primes.** Simon Stevin 27, 115–125 (1950).

Let $p_n$ be the $n$th prime, $d_n = p_n - p_{n-1}$, $z_n = n + i \log p_n$ and

$$G_N = \sum_{k=2}^{N-1} \left| \arg \frac{z_{k+1} - z_k}{z_k - z_{k-1}} \right|,$$

where $p_{n+1} \leqq N < p_{n+2}$. If $G_N$ were bounded the sequence $\log p_n$ would be convex or concave from some point onwards. Rényi has proved in a forthcoming paper that $G_N \to \infty$ as $N \to \infty$, and the present authors prove that in fact positive constants $c_2$ and $c_3$ exist such that $c_2 \log N < G_N < c_3 \log N$. Brun's method, Chebyshev's estimates for $\pi(x)$ and $p_{n+1} < 2p_n$ are all that is used. Other results proved are as follows. (a) The number of solutions of $p_n \leqq N$, $d_{n+1} = d_n + h$ does not exceed $c_{12} N (\log N)^{-\frac{1}{2}}$ [the reciprocal of this expression is given in error in the relation (25)]. (b) For any integer $N$ and any $r < c\sqrt{(\log N)}$ there is a prime $p_n \leqq N$ for which $d_{n+j} \geqq c_{13} (\log N)/r^2$ ($j = 0, 1, \cdots, r-1$). A table of $d_n$ for $n = 2, 3, \cdots, 599$ is given. [Among misprints noted the following are the more important. (i) In (12) read $d_n$ for $du$. (ii) In (14) read $(d_{n+1} - d_n)/p_n$ for $(d_{n+1} - d_n)/p_n^2$. (iii) Third last line of § 3. Read $>$ for $<$. (iv) Bottom of p. 121. Read $\underline{\lim}$ for lim.]

*R. A. Rankin* (Cambridge, England).

**Rankin, R. A. The difference between consecutive prime numbers. IV.** Proc. Amer. Math. Soc. 1, 143–150 (1950).

This paper deals with the problem of finding a lower bound for $l = \liminf_{n \to \infty} (p_{n+1} - p_n)/\log p_n$, where $p_n$ denotes the $n$th prime. It is shown that $l \leqq c(1 + 4\Theta)/5$, where $c < 42/43$ and $\Theta$ is the lower bound of all positive numbers $\sigma_0$ such that no Dirichlet function $L(s, \chi)$ has a zero at $s = \sigma + it$, where $\sigma > \sigma_0$. The method is to consider the integral $\int_0^1 |S(\theta)\Psi(\theta)|^2 d\theta$, where $S(\theta) = \sum_{N < p \leqq 2N} \exp(2\pi i \theta p) \log p$ and $\Psi(\theta)$ is a suitably defined auxiliary function. A lower bound for the integral and other necessary estimates are found by the methods of two previous papers with the same title [part II, Proc. Cambridge Philos. Soc. 36, 255–266 (1940); part III, J. London Math. Soc. 22 (1947), 226–230 (1948); these Rev. 1, 292; 9, 498].

*R. D. James* (Vancouver, B. C.).

**Wiener, Norbert, and Geller, Leonard. Some prime-number consequences of the Ikehara theorem.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 25–28 (1950).

[The second author's name is misprinted Gellert in the original.] This paper arises out of the Selberg-Erdös elementary proofs of the prime number theorem [Selberg, Ann. of Math. (2) 50, 305–313 (1949); Erdös, Proc. Nat. Acad. Sci. U. S. A. 35, 374–384 (1949); these Rev. 10, 595]. The object is to adduce reasons, from the analytical side, for the possibility of such proofs. The starting point is the identity $\zeta''/\zeta = (\zeta'/\zeta)' + (\zeta'/\zeta)^2$ in the integrated form

$$\text{(a)} \quad -\int \frac{\zeta''(z)}{\zeta(z)} dz = \int_0^\infty u^{-z} d\psi(u) + \int_0^\infty u^{-z} d\chi(u) \quad (x = \Re z > 1)$$

with $\psi(u)$ and $\chi(u)$ nondecreasing and $\psi(u) = \sum_{p_m \leqq u} \log p_m$ so that $\psi(u) \sim u$ ($u \to \infty$) is equivalent to the prime number theorem. Ikehara's theorem (in general form) gives (b) $\psi(u) + \chi(u) \sim 2u$. The possibility that $\zeta(z)$ may have zeros on $x = 1$ is irrelevant so far, since such zeros (necessarily simple) only give rise to logarithmic singularities of the function on the left of (a) and these do not affect Lebesgue integrability. But this possibility must now be excluded. If $\zeta(1 + i\lambda) = 0$, then $g(z) \equiv f(z) + \frac{1}{2}f(z + i\lambda) + \frac{1}{2}f(z - i\lambda)$ (where $f(z) = -\zeta'(z)/\zeta(z)$) is a Dirichlet series with nonnegative coefficients having no singularity at $z = 1$, from which it is "easy to show" that it has none on $x = 1$. A weak form of

Ikehara's theorem then gives:

(c) $$\int_0^u \{1+\cos(\lambda \log u)\}d\psi(u)=o(u).$$

Write $\psi(u)=\psi_1(u)+\psi_2(u)$, where $\psi_1(u)$ and $\psi_2(u)$ are non-decreasing, and increase only in those intervals where $1+\cos(\lambda \log u)\geqq\epsilon$ and $<\epsilon$ respectively. By (c), $\psi_1(u)=o(u)$ (for fixed $\epsilon>0$). By differencing (b) over intervals of variation of $\psi_2(u)$, using the fact that $\chi(u)$ is nondecreasing, and summing over the relevant intervals, it is proved that $\limsup \psi_2(u)/u\leqq\delta(\epsilon)$, where $\delta(\epsilon)$ is small with $\epsilon$. It follows that $\psi(u)=o(u)$, contrary to an elementary theorem of Chebyshev. The possibility $\zeta(1+i\lambda)=0$ is thus excluded, and the prime number theorem then follows from Ikehara's theorem, applied to $f(z)$ on classical lines. [Notes by reviewer. (i) There are several misprints and minor inaccuracies. (ii) The logical basis of the proof that $\zeta(1+i\lambda)\neq 0$ is unfortunately obscured by the words quoted. On the natural assumptions about $\zeta(z)$ we could complete the proof at once by saying that, since $g(z)$ is regular at $1+i\lambda$, therefore $\zeta(z)$ must have a pole at $1+2i\lambda$ (the classical contradiction); but this can hardly be intended, since it would render otiose all the applications of Ikehara's theorem except the classical one.] *A. E. Ingham* (Cambridge, England).

**Mikolás, Miklós. Farey series and their connection with the prime number problem. I.** Acta Univ. Szeged. Sect. Sci. Math. **13**, 93–117 (1949).

Let $\rho_\nu$ be the $\nu$th fraction of the Farey series of order $[x]$ and let $\Phi(x)$ be the number of fractions in the series. The author considers the asymptotic behavior (for $x\to\infty$) of sums of the type $\sum_{\nu=1}^{\Phi(x)}f(\rho_\nu)$, where $f(t)$ is a function defined on the interval $0<t\leqq 1$ (or at least on the rational points thereof). He proves the following theorems. (1) If $f(t)$ is (properly) Riemann-integrable in the interval $0\leqq t\leqq 1$, then $\sum_{\nu=1}^{\Phi(x)}f(\rho_\nu)\sim\Phi(x)\int_0^1 f(t)dt$. [This is problem 189 of section II of Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, v. 1, Springer, Berlin, 1925.] More generally, if $\lim_{n\to\infty}\{n^{-1}\sum_{k=1}^{n}f(k/n)\}$ exists and has the value $A$, then $\sum_{\nu=1}^{\Phi(x)}f(\rho_\nu)\sim A\Phi(x)\sim 3\pi^{-2}Ax^2$. (2) If $f(t)$ has a bounded derivative in the interval $0\leqq t\leqq 1$, then

$$\sum_{\nu=1}^{\Phi(x)}f(\rho_\nu)=\Phi(x)\int_0^1 f(t)dt+O(x\exp\{-c(\log x)^\gamma\}),$$

where $c$ is a positive constant and $\gamma$ is the same exponent that appears in the error term of the prime number theorem. With the same condition on $f(t)$ the Riemann hypothesis implies (*) $\sum_{\nu=1}^{\Phi(x)}f(\rho_\nu)=\Phi(x)\int_0^1 f(t)dt+O(x^{1+\epsilon})$. (3) Under various more stringent restrictions on the function $f(t)$, the relation (*) implies the Riemann hypothesis [cf. an earlier paper of the author, C. R. Acad. Sci. Paris **228**, 633–636 (1949); these Rev. **10**, 433]. *P. T. Bateman.*

**Mikolás, Miklós. On a theorem of J. Franel.** Norske Vid. Selsk. Forh., Trondheim **21**, no. 24, 98–101 (1949).

**Mikolás, Miklós. Un théorème d'équivalence et ses applications.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 28, 128–131 (1950).

Let $\rho_\nu(x)$ be the $\nu$th term of the Farey series $F_x$ of order $x$ when arranged in increasing order. Let $\Phi(x)=\sum_{n=1}^{[x]}\varphi(n)$ be the number of terms in $F_x$, let $\delta_\nu(x)=\rho_\nu(x)-\nu/\Phi(x)$ and let $\Delta(x)=\sum_{\nu=1}^{\Phi(x)}\delta_\nu^2(x)$. The significance of $\Delta(x)$ arises from the fact that the Riemann hypothesis is true if and only if $\Delta(x)=O(x^{-1+\epsilon})$ for each positive $\epsilon$ as $x\to\infty$. Let $M(x)=\sum_{n=1}^{[x]}\mu(n)$ and $Q(x;c,\gamma)=\exp\{-c(\log x)^\gamma\}$. Using a result of Tchudakov on the zeros of $\zeta(s)$ which implies that $\pi(x)=\operatorname{li}x+O(xQ(x;c',\gamma))$ for each $\gamma<11/21$, the author indicates that $M(x)=O(xQ(x;c,\gamma))$ in the first paper. From this he proves that $\Delta(x)=O(Q(x;c,\gamma))$.

In the second paper, the latter result is generalized as follows. Let $q(x)>0$, let there exist $q'(x)\geqq 0$ for all $x\geqq 1$ and let there be a positive constant $\eta$ such that for all sufficiently large $x$, $q(x)x^{-1+\eta}$ decreases monotonely. Then $M(x)=O(x/q(x))$ and $\Delta(x)=O(1/q^2(x))$ are equivalent to one another with a similar result holding when $O$ is replaced by $o$. In the first paper, $q(x)=1/Q(x;c,\gamma)$ and the conclusion was weaker.

It may be remarked that Tchudakov improved the result $\gamma<11/21$ to $\gamma<4/7$ in a later paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **21**, 421–422 (1938)]. In the first paper under review, the proof is incorrect in its final stages but can be fixed; the corresponding details of the second paper are likewise incorrect. By replacing the lim sup by a least upper bound, it can be shown that $\eta$ can be taken to be zero in the $O$-statement and it can be taken to be zero in the $o$-statement if it is further assumed that $q(x)x^{-\frac{1}{2}}=o(1)$.

*L. Schoenfeld* (Urbana, Ill.).

**Mikolás, Miklós. A remark concerning primitive roots of unity.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 27, 124–127 (1950).

The author gives an unnecessarily complicated proof of a result which can be reformulated and extended as follows. Let $\varphi(n)$, $\mu(n)$, and $\Lambda(n)$ denote Euler's function, the Möbius function, and the von Mangoldt function, respectively, for positive integral $n$; let the value of these functions be 0 if $n$ is not a positive integer. Let

$$F_n(z)=\prod\{z-\exp(2m\pi i/n\},\qquad 1\leqq m\leqq n,\ (m,n)=1,$$

be the cyclotomic polynomial of degree $\varphi(n)$. Let $1\neq n\neq\nu$ and $(k,\nu)=1$. If $n'=n/\nu$ then

$$F_n(e^{2k\pi i/\nu})=e^{k\pi i\varphi(n)/\nu}\prod\{2\sin\pi(m/n-k/\nu)\}$$

$$(1\leqq m\leqq n,\ (m,n)=1)$$

$$=e^{k\pi i\varphi(n)/\nu}(-1)^{k\varphi(n')}e^{\Lambda(n')}\prod_{\nu\nmid d\mid n}\{\sin dk\pi/\nu\}^{\mu(n/d)}.$$

The author's result is the one obtained from this by taking absolute values and replacing $\Lambda(n')$ by a function of $n$ and $\nu$ of more complicated appearance. This result generalizes a result of Kronecker which corresponds to the case $k=\nu=1$.

*L. Schoenfeld* (Urbana, Ill.).

**Skolem, Th. A proof of the irreducibility of the cyclotomic equation.** Norsk Mat. Tidsskr. **31**, 116–120 (1949).

This proof of the irreducibility of the polynomial $\psi_n(x)$ whose roots are the primitive $n$th roots of unity is based on two ideas; (A) If $n$ is a power of a prime $p$ then all the coefficients (except the highest) of the polynomial $(y+1)^n-1$ are divisible by $p$; (B) a form of the Schönemann-Eisenstein theorem. Let $f(x)$ be a polynomial with leading coefficient unity and with coefficients in a field $F$. If there is a prime ideal $\mathfrak{p}$ in $F$ which divides all the coefficients of $f$ (except the highest) then $f$ is irreducible in $F$ in case the constant term of $f$ is not divisible by $\mathfrak{p}^2$. The proof that $\psi_n(x)$ is not reducible in the rational field is given in detail. Kronecker's theorem that $\psi_n(x)$ is irreducible in the field generated by an $m$th root of unity, where $m$ is prime to $n$, may be dealt with in like manner.

*D. H. Lehmer* (Berkeley, Calif.).

**Richert, Hans-Egon. Über Zerlegungen in paarweise verschiedene Zahlen.** Norsk Mat. Tidsskr. **31**, 120–122 (1949).

The author generalizes some theorems by Sprague [Math. Z. **51**, 289–290, 466–468 (1948); these Rev. **10**, 283, 514] and himself [Math. Z. **52**, 342–343 (1949); these Rev. **11**, 502] about the partition of numbers into distinct parts. The general theorem may be stated as follows. Let $M$ be an infinite set of positive integers $m_1, m_2, \cdots$, with the following properties: (A) there exist integers $k \geqq 2$, $N \geqq 0$, $S > 0$, such that each integer $x$, with $N < x \leqq N+S$, may be partitioned into distinct parts taken from the first $k$ members of $M$; (B) $S \geqq m_{k+1}$; (C) $2m_i \geqq m_{i+1}$ for $i > k$. Then every integer greater than $N$ may be partitioned into distinct parts taken from $M$. The proof is quite straightforward. As an example, if $M$ is the set of triangular numbers then (A), (B) and (C) are satisfied with $k=8$, $N=33$, $S=45$. Hence every integer except 2, 5, 8, 12, 23, 33 is the sum of distinct triangular numbers. *D. H. Lehmer.*

**Ostmann, Hans-Heinrich. Verfeinerte Lösung der asymptotischen Dichtenaufgabe.** J. Reine Angew. Math. **187**, 183–188 (1950).

Let $\mathfrak{A}$ be a set of nonnegative integers, $A(n)$ the number of positive elements of $\mathfrak{A}$ not exceeding $n$, and let $\delta^*(A) = \liminf_{n\to\infty} A(n)/n$ be the asymptotic density of $\mathfrak{A}$. Let $\mathfrak{A}+\mathfrak{B}$ be the set of all numbers $a+b$ where $a \epsilon A$ and $b \epsilon B$. The author proves the following result: If $\mathfrak{B}$ contains $k \geqq 1$ consecutive natural numbers, then

$$\delta^*(\mathfrak{A}+\mathfrak{B}) \geqq \delta^*(\mathfrak{A}) + (1-k^{-1})\delta^*(\mathfrak{B}).$$

This represents an extension of a result of Erdős [Ann. of Math. (2) **43**, 65–68 (1942); these Rev. **3**, 165] who showed that if $0 \epsilon \mathfrak{A}$, if $0, 1 \epsilon \mathfrak{B}$, and if $\delta^*(\mathfrak{B}) \leqq \delta^*(\mathfrak{A})$, then $\delta^*(\mathfrak{A}+\mathfrak{B}) \geqq \delta^*(\mathfrak{A}) + \frac{1}{2}\delta^*(\mathfrak{B})$; the author also proves this result without the assumption that $\delta^*(\mathfrak{B}) \leqq \delta^*(\mathfrak{A})$.

*L. Schoenfeld* (Urbana, Ill.).

**Bochner, S., and Chandrasekharan, K. Summations over lattice points in $\kappa$-space. (A supplementary note.)** Quart. J. Math., Oxford Ser. (2) **1**, 80 (1950).

Corrections of some errors and misprints occurring in a previous paper with the same title [Quart. J. Math., Oxford Ser. (1) **19**, 238–248 (1948); these Rev. **10**, 431].

*N. G. de Bruijn* (Delft).

**Teghem, J. Sommes de Weyl. Sur la méthode de Vinogradow–van der Corput.** Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **23**, no. 5, 50 pp. (1949).

This paper is devoted to the detailed proof of three theorems on the so-called Weyl sums $S = \sum_{x=Q+1}^{Q+X} \exp\{2\pi i f(x)\}$. Here $f(x)$ is a real polynomial of degree $k \geqq 2$ or, more generally, a real function on the integers whose $(k+1)$th difference is small, $X$ is an integer not less than $k+2$, and $Q$ is any integer whatever. The theorems give improved estimates of $S$, the statements of which are too long to quote here. The work follows the lines of van der Corput's form of the method of Vinogradov [cf. Mathematica, Zutphen. B. **5**, 1–30 (1936)]. The author has already indicated the applications of the results of the present memoir [C. R. Congrès Sci. Math. Liége, 1939, pp. 84–88; Bull. Soc. Roy. Sci. Liége **11**, 4–6 (1942); Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 593–603 (1948); these Rev. **7**, 146; **10**, 354]. *P. T. Bateman* (Urbana, Ill.).

**Perron, Oskar. Neuer Beweis zweier klassischer Sätze über Diophantische Approximationen.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 125–130 (1950).

A well-known theorem of Hurwitz states that any irrational number $\theta$ has infinitely many rational approximations $x/y$ $(y>0)$ such that $y|\theta y - x| < 5^{-\frac{1}{2}}$. A closely related theorem of Korkine and Zolotareff states that a quadratic form $ax^2+bxy+cy^2$ with $b^2-4ac=1$ assumes a value numerically not exceeding $5^{-\frac{1}{2}}$. The author gives simple proofs of these theorems. The proof is on the same lines in each case, and uses the lemma that if $0<\delta<1$ and $0<\lambda<1$ then one at least of the three numbers $\delta$, $\lambda+\delta\lambda^2$, $1-\lambda-\delta(1-\lambda)^2$ does not exceed $5^{-\frac{1}{2}}$. *H. Davenport* (London).

**Perron, Oskar. Ein Beweis für die Primalität der Zahl $2^{31}-1 = 2\,147\,483\,647$.** S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 223–226 (1949).

The number $N = 2^{31}-1$ was proved prime by Euler. The present indirect proof is a shorter one and depends essentially on showing the nonexistence of a nontrivial representation of $N$ as the difference of squares. No use is made of a list of primes. The author's remarks about positive tests for primality are, in the opinion of the reviewer, unduly disparaging. *D. H. Lehmer.*

**Jarden, Dov, and Katz, Alexander. Additional page (477) to D. N. Lehmer's Factor Table.** Riveon Lematematika **3**, 49, 52 (1 plate) (1949). (Hebrew. English summary)

This note gives a one-page extension of Lehmer's Factor Table for the First Ten Millions [Carnegie Institution of Washington, Washington, D. C., 1909], thus extending the table to 10038000. There is also an account of other factor tables beyond $10^7$.

*D. H. Lehmer* (Berkeley, Calif.).

## ANALYSIS

### *Theory of Sets, Theory of Functions of Real Variables*

**Sierpiński, Waclaw. Solution de l'équation $\omega^\xi = \xi^\omega$ pour les nombres ordinaux.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 49–50 (1950).

The author solves a special case of the problem of finding distinct ordinal numbers $\alpha$ and $\beta$ which satisfy the equation $\alpha^\beta = \beta^\alpha$. The general problem, however, has already been completely solved by E. Jacobsthal [Math. Ann. **64**, 475–488 (1907), especially § 4]. *F. Bagemihl.*

**Cuesta, N. Ascending sequences of ordinal numbers.** Revista Mat. Hisp.-Amer. (4) **9**, 83–96, 168 (1949). (Spanish)

Let $M$ be the set of all ascending sequences of type $\omega_1$, whose terms are ordinal numbers of $Z(\aleph_0)$. The author writes $a <'' b$, where $a, b \epsilon M$, to signify that eventually the terms of $a$ are less than the respective terms of $b$. The set $M$ is partially ordered by the relation $<''$, and its maximal fully ordered subsets are well-ordered and of type $\geqq \omega_2$. The basic idea of Eyraud in attempting to prove the continuum hypothesis was to calculate the number of distinct segments of the elements of a fully ordered subset, of type $\omega_2$,

of $M$. The author maintains that Eyraud included sequences whose identity with such segments is problematical and as difficult to establish as the continuum hypothesis itself, and that by following Eyraud's arguments [C. R. Acad. Sci. Paris **224**, 85–87 (1947), especially lines 8–12 on p. 87; these Rev. **8**, 448] one can also arrive at the false result that the continuum is enumerable. Finally, the author offers, as a possible explanation of the difficulty of the continuum problem, the analogy between Cantor's ordinal numeration and the Roman system for finite ordered sets, and the difficulties which the second presents in finite arithmetic. (Two corrections are listed on p. 168, the first of which should refer to theorem 3.) *F. Bagemihl.*

**Novák, J. On a problem of E. Čech.** Proc. Amer. Math. Soc. **1**, 211–214 (1950).

Let $P$ be any set, and let $\mathfrak{P}$ be the family of all subsets of $P$. Let $u$ be a single-valued function with domain $\mathfrak{P}$ and range contained in $\mathfrak{P}$ such that (a) $u(0)=0$, (b) $u(M)\supset M$ for all $M\in\mathfrak{P}$, (c) $u(M\cup N)=u(M)\cup u(N)$ for all $M, N\in\mathfrak{P}$. Let $u^0(M)=M$, $u^1(M)=u(M)$, $u^\xi(M)=u(u^{\xi-1}(M))$ for non-limit ordinals $\xi$, and let $u^\xi(M)=\bigcup u^\eta(M)$, $0\leqq\eta<\xi$, for limit ordinals $\xi$. Let $\varphi(M)$ be the least ordinal such that $u^{\xi+1}(M)=u^\xi(M)$. The following theorem is proved. Let $H$ be a nonvoid set of ordinal numbers. Then there exists a countable set $P$ and a set function $u$ on $\mathfrak{P}$ satisfying (a)–(c) above such that $H$ is exactly the set of all $\varphi(M)$ for $M\in\mathfrak{P}$ if and only if the following conditions are satisfied: (1) $H$ contains only countable ordinals and contains 0 and 1; (2) $\alpha+\beta\in H$ implies that $\beta\in H$. This theorem solves a problem proposed by E. Čech [Časopis Pěst. Mat. Fys. **66**, D225–D264 (1937)]. *E. Hewitt* (Seattle, Wash.).

**Stone, M. H. Algebraic formulation of the problem of measure.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 69–74 (1950).

Le "problème de la mesure" tel qu'il est défini par Tarski [Fund. Math. **31**, 47–66 (1938)] est le suivant: étant donné un ensemble $X$ et une relation $R(x, y, z)$ entre trois éléments génériques de $X$, une "mesure $R$-additive" est une fonction numérique $\mu\geqq 0$ définie dans $X$, et telle que la relation $R(x, y, z)$ entraîne $\mu(x)=\mu(y)+\mu(z)$. En plongeant $X$ dans l'espace vectoriel des combinaisons linéaires formelles d'éléments de $X$ à coefficients rationnels l'auteur ramène ce problème au suivant: étant donné un espace vectoriel $V$ sur le corps des rationnels, et une partie non vide $V_0$ de $V$, trouver une fonction numérique $\lambda$ définie dans $V$, telle que $\lambda(u+v)=\lambda(u)+\lambda(v)$, $\lambda(u)\geqq 0$ pour $u\in V_0$, et $\lambda(v_0)>0$ pour un élément $v_0\in V_0$. Il montre que la condition nécessaire et suffisante pour qu'il existe une telle fonction est que $-v_0$ soit point interne d'un ensemble convexe $K$ engendrant $V$ et ne rencontrant pas le plus petit cone convexe contenant $V_0$. *J. Dieudonné* (Nancy).

**Riesz, F. L'évolution de la notion d'intégrale depuis Lebesgue.** Ann. Inst. Fourier Grenoble **1** (1949), 29–42 (1950).

A lecture outlining the history of Lebesgue integration and the author's contributions to the subject. *P. R. Halmos* (Chicago, Ill.).

**Riesz, Frigyes. L'évolution de la notion d'intégrale de Lebesgue.** Mat. Lapok **1**, 79–90 (1950). (Hungarian. Russian and French summaries)

Hungarian version of the paper reviewed above.

**Denjoy, Arnaud. La topologie des fonctions.** C. R. Acad. Sci. Paris **226**, 1407–1410 (1948).

The results of this paper are similar to those of the paper reviewed below, where an explicit theorem is formulated. The author merely lists some very general hypotheses (but states no theorem) under which the arguments on pp. 151–158 of the second part of his Leçons sur le calcul des coefficients d'une série trigonométrique [tome II, Gauthier-Villars, Paris, 1941; these Rev. **8**, 260] remain valid. *H. Busemann* (Los Angeles, Calif.).

**Choquet, Gustave. Sur un théorème récent de M. Denjoy.** C. R. Acad. Sci. Paris **226**, 1670–1672 (1948).

Let $X$ and $Y$ be metric spaces and $A$ the graph in $X\times Y$ of a continuous mapping of $X$ in $Y$. Let $B\subset(X\times Y)-A$. For $a\in A$ let $[a]$ be the set in $X\times Y$ with the same first coordinate as $a$. For every $b\in B$ let $\delta(b)$ be a closed, possibly empty, subset of a separable metric space $\Delta$. For every neighborhood $V$ of $a\in A$ in $X\times Y$ denote by $C_V(a)$ and $P_V(a)$ the closures of the respective unions $\bigcup\delta(b)$, $b\in(B\cap V\cap[a])$, and $\bigcup\delta(b)$, $b\in(B\cap V)$, and put $C(a)=\bigcap_V C_V(a)$, $P(a)=\bigcap_V P_V(a)$. Then $C(a)\subset P(a)$ and $P(a)$ is upper semicontinuous. (The definitions of $C(a)$ and $P(a)$ are suggested by the concepts of contingent and paratingent of Bouligand.) The following theorem holds. If there exists a countable base $V_1, V_2, \cdots$ for a neighborhood of $A$ in $X\times Y$ such that for each $i$ the function $C_{V_i}(a)$ is semicontinuous in $a$, then the set of points of $A$ for which $C(a)\neq P(a)$ is the union of a countable number of sets nowhere dense in $A$. *H. Busemann.*

**Dilworth, R. P. The normal completion of the lattice of continuous functions.** Trans. Amer. Math. Soc. **68**, 427–438 (1950).

Let $S$ be a completely regular space, let $x\in S$, and let $N_x$ be an open set containing $x$. If $\phi$ is a bounded real function on $S$, define $\phi^*$ and $\phi_*$ by $\phi^*(x)=\inf_{N_x}\sup_{y\in N_x}\phi(y)$, $\phi_*(x)=\sup_{N_x}\inf_{y\in N_x}\phi(y)$. Then $\phi$ is called normal upper semicontinuous if $\phi^*=\phi=(\phi_*)^*$. The author's first main result is that the normal completion $\bar{C}(S)$ of the lattice $C(S)$ of real bounded continuous functions on $S$ is isomorphic to the lattice of all normal upper semicontinuous functions on $S$. The other main result states that $\bar{C}(S)$ is isomorphic to the lattice of all continuous functions on the Boolean space associated with the Boolean algebra of regular open sets of $S$. *S. B. Myers* (Ann Arbor, Mich.).

**Rényi, Alfréd. On the measure of equidistribution of point sets.** Acta Univ. Szeged. Sect. Sci. Math. **13**, 77–92 (1949).

Let $E$ be a set on $(0, 1)$, of measure $|E|$, and characteristic function $F(x)$; $F(x)$ is extended so as to be periodic with period 1. Let $G(t)=\int_0^1 F(x)F(x+t)dx$, $m(E)=\min G(t)$, $\mu(E)=m(E)/|E|^2$, so that $0\leqq\mu(E)<1$ if $0<|E|<1$. The author calls $\mu(E)$ the measure of equidistribution of $E$; it actually measures the equidistribution of the set of distances between points of $E$. The main purpose of the paper is to construct a set $E$ with assigned (arbitrarily small) positive measure and $\mu(E)$ arbitrarily close to 1. Such a set always intersects all its translates in a set whose measure is bounded from zero. The author then introduces a measure of $k$-fold equidistribution and proves a weaker theorem for $k>1$ which, however, implies a generalization of this intersection property. Finally he makes a connection with Singer's "difference bases" in number theory and proves a theorem about them. *R. P. Boas, Jr.* (Evanston, Ill.).

**Kuipers, L., and Meulenbeld, B. Some theorems in the theory of uniform distribution.** Nederl. Akad. Wetensch., Proc. **53**, 305–308 = Indagationes Math. **12**, 53–56 (1950).

Generalising one of their previous results [same Proc. **52**, 1158–1163 = Indagationes Math. **11**, 432–437 (1949); these Rev. **11**, 423] the authors prove the following result. Let $f(t)$ be differentiable, $t \geqq 0$, and assume that $tf'(t)$ is bounded and that for $t > T^*$, we have $|tf'(t) - A| < B < \frac{1}{2}\pi^{-1}$, where $A$ and $B$ are fixed numbers; then $f(t)$ is not $C^1$ uniformly distributed (mod 1). *P. Erdös* (Aberdeen).

**Meulenbeld, B. On the uniform distribution of the values of functions of $n$ variables.** Nederl. Akad. Wetensch., Proc. **53**, 311–317 = Indagationes Math. **12**, 59–65 (1950).

The author extends the definition of $C^1$ uniform distribution to functions of $n$ variables and gives various conditions for a function in $n$ variables to be $C$-uniformly distributed. For the definition of $C^1$ uniform distribution see Kuipers and the author [same Proc. **52**, 1151–1157 = Indagationes Math. **11**, 425–431 (1949); these Rev. **11**, 423]. *P. Erdös* (Aberdeen).

**Kronrod, A. S. On functions of two variables.** Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 24–134 (1950). (Russian)

This memoir gives an introduction to what the author calls "the geometrical theory of real functions of two variables." He deals chiefly with continuous functions $F(\eta)$ of a point $\eta$ in a two-dimensional region $J$ (either a square or a 2-sphere) and introduces concepts designed to be invariant either under homeomorphic transformation of $J$ onto itself (e.g., linear variation, monotony relative to a point) or under change of coordinate axes (e.g., planar variation, "line integral," "derivative"). He finds it helpful to illustrate concepts of the former type by means of the one-dimensional tree whose points correspond to the components of the level-sets of $F(\eta)$.

The principal definitions and (new) results of the memoir were stated (without the details and proofs now supplied) in two previous notes [Doklady Akad. Nauk SSSR (N.S.) **66**, 797–800, 1041–1044 (1949); these Rev. **11**, 19, 90]. Accordingly, it will suffice here to remark that the author presents his theory of "line integrals" and "derivatives" with rather less detail than the rest because it has less appearance of finality, and that it is connected with his theory of linear variation by the following result (theorem 38). With the terminology of the reviews cited, if $F(\eta)$ is a twice differentiable function and $f(\eta)$ is its "derivative" then the "line integral" of $|f(\eta)|$ from $\xi$ to $\zeta$, where $\xi, \zeta$ are any two points of $J$, gives the linear variation of $F(\eta)$ "from $\xi$ to $\zeta$" (i.e., counting only those components of level-sets that separate $\xi$ and $\zeta$). *H. P. Mulholland.*

**Rado, T., and Reichelderfer, P. V. On generalized Jacobians.** Trans. Amer. Math. Soc. **68**, 405–419 (1950).

Given a function $f(w)$, $w = (u, v)$, $w \varepsilon R$, $R$ a simple Jordan region on the $(u, v)$-plane $E_2$, then $f(w)$ is said to have an approximate limit $\alpha$ at a point $w_0 \varepsilon R_0$, $R_0$ interior of $R$, if for any $\epsilon > 0$, $w_0$ is a point of density of the set of the points $w$ such that $|f(w) - \alpha| \leqq \epsilon$. If $f^*(w_0) = \alpha$ then, under general conditions for $f(w)$, the function $f^*(w)$ is defined almost everywhere (a.e.) in $R_0$ and $f^*(w) = f(w)$ a.e. in $R_0$. Given in the Euclidean space $E_3$ a surface $S$ of finite Lebesgue area and parametrically represented on $R$, then the derivative $D(w)$ of the Lebesgue area as a rectangle function is defined a.e. in $R_0$. If $X = Ox^1x^2x^3$, $Y = Oy^1y^2y^3$ are any two right handed orthogonal systems of coordinates in $E_3$ then we have two different parametrical representations of $S$ on $R$ related by elementary linear relations. We have also two groups $j(w) = (j_1, j_2, j_3)$, $J(w) = (J_1, J_2, J_3)$ of generalized Jacobians, each defined a.e. in $R_0$, i.e., for any $w \varepsilon R_0 - e(X)$, $w \varepsilon R_0 - e(Y)$, where $e(X)$, $e(Y)$ are sets of measure zero. We also know that (i) $j(w)$ and $J(w)$ are related by the same elementary linear relations which hold for the coordinates a.e. in $R_0$, i.e., for any $w \varepsilon R_0 - e(X, Y)$, $e$ a set of measure zero [Cesari, Comment. Math. Helv. **22** (1949), 1–16 (1948); Reichelderfer, Duke Math. J. **16**, 73–83 (1949); these Rev. **10**, 109; **11**, 20], and (ii) $D(w) = |j(w)|$ a.e. in $R_0$ [Cesari, loc. cit.]. The problem arises if the existence of the generalized Jacobians and the relations (i) and (ii) can be stated "uniformly a.e. in $R_0$," i.e., for any $w \varepsilon R_0 - e$, where $e$ is a set of measure zero independent of $X$ or $Y$. The main result of the paper is that this aim is attained if we consider instead of $j(w)$, or $J(w)$, the approximate limits $j^*(w)$, or $J^*(w)$. *L. Cesari* (Lafayette, Ind.).

**Pucci, Carlo. Un teorema di derivazione per serie.** Boll. Un. Mat. Ital. (3) **5**, 20–23 (1950).

A correction to a previously published proof on termwise differentiation of series [Boll. Un. Mat. Ital. (3) **4**, 270–274 (1949); these Rev. **11**, 428]. *A. Zygmund.*

**Picone, Mauro. Sulla derivazione parziale per serie.** Boll. Un. Mat. Ital. (3) **5**, 24–33 (1950).

Results obtained by Pucci [cf. the preceding review] are extended to partial differentiation of series whose terms are functions of several variables, or even vectors. The author constantly uses the notion of the "perimetric diameter" of an open domain $X$. It is the upper bound of the numbers $\pi(x', x'')$, where $x' \varepsilon X$, $x'' \varepsilon X$, and $\pi(x', x'')$ denotes the lower bound of the lengths of the polygonal lines joining $x'$ and $x''$, and having all sides parallel to the coordinate axes. *A. Zygmund* (Cambridge, Mass.).

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### *Theory of Functions of Complex Variables*

**Wagner, Karl Willy. Über den Zusammenhang von Amplituden- und Phasenverzerrung.** Arch. Elektr. Übertragung **1**, 17–28 (1947).

**Pleijel, Henning. Auspaltung einer analytischen Funktion in einen geraden und einen ungeraden Teil, sowie ein Verfahren, um bei bekannter gerader Komponente die ungerade, und die Funktion als Ganzes zu bestimmen.** Arch. Elektr. Übertragung **2**, 307–320 (1948).

The engineering importance of the (loss-phase) relation between the real and imaginary parts of a network function with no singularities in the right half $p$-plane is well known [see Y. W. Lee, J. Math. Physics **11**, 83–113 (1932); H. W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand, New York, 1945]. Wagner's paper rederives some of the ideas of these references. Then an application is made to an ideal filter (whose transfer function is $\exp(-ia\omega)$ for $|\omega| < \Omega$ and 0 if $|\omega| > \Omega$). The real part of this function is used to compute an imaginary part and the resulting complex function is found to approximate an ideal filter characteristic as the delay $a$ increases.

Pleijel treats the same problem from the point of view of finding a function $f(p)$ which is regular in the right half plane and with even part equal to a prescribed even

function $f_1(p)$. If one succeeds in finding an $A(p)$ with no singularities in the right half plane and such that $f_1(p)=A(p)+A(-p)$, then $f(p)=2A(p)$. If $f_1$ is rational, $A$ is obtained immediately by a partial fraction expansion of $f_1$. A more difficult example is also discussed in which $f_1$ is the square root of a rational function.

*E. N. Gilbert* (Murray Hill, N. J.).

**Zin, G. La continuazione analitica delle funzioni associate ai sistemi fisici lineari.** Nuovo Cimento (9) **6**, 531–542 (1949).

The value of a continuous function $f(x)$ at an interior point $x_0$ of the interval $[-1, 1]$ can be expressed as the limit of integrals taken over $[-1, 1]$. In case $f$ is analytic in the upper half plane, the author obtains a similar expression for $f(x_0)$ when $x_0$ is exterior to $[-1, 1]$. A mapping is made which sends $[-1, 1]$ into a semicircle $C$ from $-1$ to 1, and $x_0$ into a real point $u_0$ on $[-1, 1]$. Then $C$ is replaced by $[-1, 1]$ and the original representation is applied.

*R. C. Buck* (Madison, Wis.).

**Kössler, Miloš. The signification of the number** $\sup |a_n|^{\frac{1}{n}}$ **in the theory of power series.** Časopis Pěst. Mat. Fys. **74**, 47–53 (1949). (Czech. English summary)

The author deduces a series of theorems which show how important for power series is the number $\sup |a_n|^{1/n}$. Using this number, a best possible lower bound is given for the smallest zero of $\sum_0^\infty a_n z^n$ and $\sum_0^\infty a_{nm} y^n z^m$. The result is generalized in a significant way. Further, a best possible lower bound is given for the radius of the inverse series and of the circle in which the given function is schlicht.

*František Wolf* (Berkeley, Calif.).

**de Castro Brzezicki, Antonio. On the analytic continuation of Dirichlet series.** Revista Acad. Ci. Madrid **43**, 139–166 (1949). (Spanish)

Theorems related to the Fatou-Pólya theorem on Taylor series. Generalisations of such theorems (theorem of Pólya, theorem of Mandelbrojt) are given for Dirichlet series. The greater part of the paper is expository.

*S. Mandelbrojt* (Houston, Tex.).

**Džrbašyan, M. M. On the completeness of an orthogonal system of entire periodic functions.** Doklady Akad. Nauk SSSR (N.S.) **71**, 429–432 (1950). (Russian)

Let $\sigma, \omega>0$, $L_0(z)=2\sigma/\pi$,

$$L_n(z)=(2\sigma/\pi)\prod_{k=0}^{n-1}(e^{\pi z/\omega}-e^{\pi^2 k/2\omega^2}),$$

$\phi_n(z)=L_n(z)/(L_n(\pi n/2\sigma\omega))^{\frac{1}{2}}$. Then

$$\int_0^\infty\int_0^{2\pi}\phi_n(re^{i\theta})\overline{\phi_m(re^{i\theta})}\exp(-2\sigma r^2)r\,dr\,d\theta=\delta_{nm}.$$

If $f(z)$ is an integral periodic function of period $2\omega$ with

$$\int_0^\infty\int_0^{2\pi}|f(re^{i\theta})|^2\exp(-2\sigma r^2)r\,dr\,d\theta<\infty$$

and

$$A_n=\int_0^\infty\int_0^{2\pi}f(re^{i\theta})\overline{\phi_n(re^{i\theta})}\exp(-2\sigma r^2)r\,dr\,d\theta,$$

then $\sum_{k=0}^\infty A_k\phi_k(z)$ converges uniformly to $f(z)$ in any finite part of the $z$-plane. The $n$th partial sum of the series is identical with the interpolating "polynomial" $g_n(z)=\sum_{k=0}^n B_k e^{\pi k z/\omega}$ satisfying $g_n(\pi k/2\sigma\omega)=f(\pi k/2\sigma\omega)$ $(k=0, 1, 2, \cdots, n)$.

*W. H. J. Fuchs* (Ithaca, N. Y.).

**Belardinelli, G. Convergenza ed ultraconvergenza delle serie di interpolazione.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 99–147 (1949).

L'auteur démontre pour les séries de la forme

$$\sum C_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

quelques théorèmes qui généralisent ceux de Hadamard et Pólya (séries non prolongeables) d'Ostrowski (ultraconvergence), ainsi que le théorème d'Abel.

*S. Mandelbrojt.*

**Belardinelli, G. Convergenza di alcune serie di funzioni.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 329–346 (1949).

L'auteur étudie la convergence des séries de la forme $\sum c_n(z-\alpha_1)\cdots(z-\alpha_n)/(z-\beta_1)\cdots(z-\beta_n)$. Il détermine la région de convergence absolue de telles séries. Ces recherches généralisent les résultats de R. Lagrange [Acta Math. **64**, 1–80 (1935)].

*S. Mandelbrojt* (Houston, Tex.).

**Rényi, Alfréd. On the geometry of conformal mapping.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 215–222 (1950).

Let $f(z)=z+a_2z^2+\cdots$ be univalent in the unit circle and let $C(r)$ denote the conformal image of the circumference $|z|=r$ $(0<r<1)$ which is yielded by the mapping $z\to f(z)$. By suitable utilization of known results on univalent functions in the unit circle, the author obtains additional geometric information regarding the curves $C(r)$. For instance, he proves that, for small $r$, $C(r)$ is contained between two circles of radii $r+O(r^2)$ and $r-O(r^2)$, respectively.

*Z. Nehari* (St. Louis, Mo.).

**Epheser, Helmut. Konforme Abbildung einfach zusammenhängender Gebiete, die von Bögen konzentrischer logarithmischer Spiralen berandet sind.** J. Reine Angew. Math. **187**, 131–152 (1950).

The mapping $z\to e^z$ transforms a polygon in the complex $z$-plane into a domain $D$ bounded by logarithmic spirals whose asymptotic point is at the origin; obviously, $D$ contains neither the origin nor the point at infinity. Such domains can therefore be mapped onto the unit circle by means of the Schwarz-Christoffel formula. The author shows that by a slight modification of the latter formula it is also possible to treat the cases in which $D$ contains the origin, the point at infinity, or both.

*Z. Nehari.*

**Narodeckii, M. Z., and Šerman, D. I. On a problem of conformal mapping.** Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 209–214 (1950). (Russian)

Let $S$ be the doubly-connected region bounded on the inside by a circle of radius $R$, and on the outside by a concentric ellipse with semi-axes $a$ and $b$, $a>b$. The function $\varphi(z)$ which maps $S$ conformally onto the region $\rho_1>|w|>\rho_2$ is obtained in the form of an infinite series. A small number of terms of the series may be used to give a good approximation to $\varphi(z)$ when $c=R/b$ is a small number. Detailed computations are given for $c=.93$, $k=a/b=1.25$, using 5 terms of the series, and for $c=.71$, $k=3$, using 7 terms of the series. The relative error on the boundary of the ellipse is on the order of .003. It is stated that the same method used to determine $\varphi(z)$ can be applied to determine the mapping function for a number of other elementary doubly-connected regions.

*A. W. Goodman* (Lexington, Ky.).

**Egerváry, E. On the mapping of the unit-circle by polynomials.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 226–230 (1950).

The author considers the set Π of polynomials $P$ of degree not exceeding $n$, with $P(0)=0$ and real part not less than $-1$ for $|z|\leqq 1$. Fejér [J. Reine Angew. Math. 146, 53–82 (1913)] proved that $-1\leqq \Re P(e^{i\theta})\leqq n$, while Szász [Math. Z. 1, 149–162 (1918)] proved that $|\Im P(e^{i\theta})|\leqq \cot \frac{1}{2}\pi/(n+1)$. The author extends these results by determining the precise region of points covered by the map of the unit circle by at least one $P\in\Pi$: it is the convex cover of the map by $P^*=2(n+1)^{-1}\{nz+(n-1)z^2+\cdots+z^n\}$; its supporting function with respect to 0 is $\sin\{n\theta/(n+1)\}\csc\{\theta/(n+1)\}$. Furthermore, the eccentricity with respect to $Q(0)$ of the map of $|z|=1$ by an arbitrary polynomial $Q(z)$ of degree $n$ does not exceed $n$, and the extremum is attained for $a+bP^*(\epsilon z)$, $|\epsilon|=1$, eccentricity of a set being defined as the ratio of the maximum and minimum of the supporting function of its convex cover. *R. P. Boas, Jr.*

**Lavrent'ev, M. A. A fundamental theorem of the theory of quasi-conformal mapping of plane regions.** Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 513–554 (1948). (Russian)

The author calls a homeomorphic mapping (1) $u=u(x, y)$, $v=v(x, y)$ transforming a domain $D$ into a domain $\Delta$ where $u$ and $v$ satisfy the system (2) $\Phi_\kappa(x, y, u, v, u_x, u_y, v_x, v_y)=0$, $\kappa=1, 2$, $u_x=\partial u/\partial x$, $\cdots$, a quasi-conformal transformation corresponding to the system (2). Let $Q$ be a square in the $(u, v)$-plane with the sides $w_0w_1$, $w_0w_2$, and let $\nu$ be the angle which $w_0w_1$ forms with the $u$-axis. By the transformation (1), $Q$ goes into a parallelogram $z_0z_1z_2z_3$. Let $z_2-z_0=V_\nu e^{i\omega_\nu}$, $\theta_\nu=\arg[(z_2-z_0)/(z_1-z_0)]$, $W_\nu V_\nu\Delta_1=1$, where $\Delta_1=u_xv_y-u_yv_x$. The system (2) can be replaced by the relations

(3) $W_\nu=F_1^{(\nu)}(V_\nu, \alpha_\nu; x, y, u, v)$, $\theta_\nu=F_2^{(\nu)}(V_\nu, \alpha_\nu; x, y, u, v)$.

The author calls the system (3) "strongly elliptic" if (i) $F_1^{(\nu)}$ and $F_2^{(\nu)}$ are single-valued and continuous; (ii) there exists a constant $k$, $k>0$, such that for all values of the arguments of $F_1^{(\nu)}$ and $F_2^{(\nu)}$ we have $k<\theta<\pi-k$, $\Delta_1>0$; (iii) for any fixed $\alpha_\nu$, $x$, $y$, $u$, $v$ the function $F_1^{(\nu)}$ increases monotonically with respect to $V_\nu$, $V_\nu>0$, and we have $(\partial F_1^{(\nu)}/\partial V^{(\nu)})>k>0$. In the present paper the following result is stated. Let $D$ and $\Delta$ be two domains bounded by piecewise smooth curves, and let $(z_1, z_2, z_3)$ and $(w_1, w_2, w_3)$ be two sets of three boundary points of $D$ and $\Delta$, respectively. If $F_1^{(\nu)}$, $F_2^{(\nu)}$ are strongly elliptic and possess uniformly continuous partial derivatives, then there exists a unique quasi-conformal transformation (1) corresponding to (3), which transforms $D$ into $\Delta$, and such that the points $z_\kappa$ go into $w_\kappa$, $\kappa=1, 2, 3$. The consideration proceeds in several steps. At first it is shown that instead of the above mapping, one can consider a quasi-conformal mapping of a curvilinear strip in the $(y, u)$-plane into the strip $0<v<1$ in the $(x, v)$-plane, so that $\pm\infty$ go into $\pm\infty$, respectively, and where $y(x, v)$, $u(x, v)$ satisfy a system of two nonlinear equations. The problem of the solution of the above system can be replaced by solutions of the equation

(4) $y_{vv}-\alpha_1y_{xv}-\alpha_2y_{xx}=\alpha_3$, $\alpha_\kappa=\alpha_\kappa(y, u, y_v, y_x, x, v)$,

$\kappa=1, 2, 3$, under boundary conditions: $y(x, 0)=Y_1(x)$, $y(x, 1)=Y_2(x)$. [See also Lavrent'ev, Mat. Sbornik N.S. 21(63), 285–320 (1947); these Rev. 10, 290.]

The special case where $\alpha_\kappa$, $\kappa=1, 2, 3$, are independent of $y$ and $u$ is first considered. Due to the "strong ellipticity" of the basic system, (4) is elliptic, i.e., $-4\alpha_2-\alpha_1^2>0$. (The author also makes some additional assumptions which he gets rid of subsequently.) The strip $0<v<1$ is divided into $n$ strips $S_i$: $((i-1)/n)<v<(i/n)$, $i=1, 2, \cdots, n$, and the author introduces an approximate solution $Y_n(x, v)$ which in each strip $S_i$ satisfies the equation $y_{vv}-\alpha_{1,n}^{(0)}y_{xv}-\alpha_{2,n}^{(0)}y_{xx}=\alpha_{3,n}^{(0)}$ where $\alpha_{k,n}^{(0)}$ approximate $a_k$, $k=1, 2, 3$. He determines bounds for the variation of $Y_n(x, v)$ when the boundary values vary, and states that the $Y_n(x, v)$, $n=N, N+1, \cdots$ form a normal family, from which the existence of the solution of (4) (in the special case under consideration) follows. He next considers the general case, returning to the system (3). From the results in the special case it follows that a quasi-conformal mapping corresponding to a system

(5) $W=F_1(V, \alpha, x, a_2^{(n)}(X), u^{(n)}(X), v)$,
$\theta=F_2(V, \alpha, x, a_2^{(n)}(X), u^{(n)}(X), v)$,

exists. Dividing the strip $0<v<1$ into $n$ strips $S_i$, and using the results previously obtained he constructs an approximate solution of (3) in every strip $S_i$, and shows that for $n=N, N+1, \cdots$ these approximate solutions form a normal family, from which the existence of a unique solution of the original equation (3) is concluded. In order to prove these results a number of theorems on quasi-conformal transformation, e.g., an analogue of the Schwarz-Lindelöf principle, are used. It would be desirable for the author to present some of his considerations in more detail.

*S. Bergman* (Cambridge, Mass.).

**Hervé, Michel. Sur quelques problèmes d'extremum relatifs aux fonctions analytiques et uniformes sur un domaine doublement connexe.** C. R. Acad. Sci. Paris 230, 609–610 (1950).

Let $D$ be a doubly-connected domain in the complex $z$-plane and let $z=\phi(Z)$ denote the function mapping $|Z|<1$ onto the universal covering surface of $D$. If the linear substitutions which leave $\phi(Z)$ invariant are denoted by $S$, the author considers the class $\mathfrak{H}(\theta)$ of functions $H(Z)$ which are regular and bounded in $|Z|<1$ and satisfy $H(SZ)=e^{-i\theta}H(Z)$, where $\theta$ is a given angle. The extremal problem $|H(Z_0)|=\text{maximum}$, $H\leqq\mathfrak{H}(\theta)$, is posed and a complete characterization of its solutions is given.

*Z. Nehari* (St. Louis, Mo.).

**Markuševič, A. I. Some problems of the theory of boundary properties of analytic functions.** Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 3–18 (1949). (Russian)

This paper is a review of facts, problems and questions connected with the boundary values of a function analytic in $|z|<1$. Different kinds of boundary values at a boundary point are considered: in the full neighborhood, in a sector and radial limits. The special points touched on are exemplified by the following: (a) Fatou's theorem with counterexamples showing the necessity of its assumptions, (b) examples of functions with different singular behavior at the boundary, (c) functions of bounded type, (d) Plessner's theorem [J. Reine Angew. Math. 158, 219–227 (1927)] asserting that an analytic function, whose boundary values at every point of a set $S$ of positive measure in a sector are not dense in the whole plane, has definite boundary values in the sector almost everywhere in $S$, (e) characterization of a boundary function of an analytic function and the construction of the latter from the former.

*František Wolf* (Berkeley, Calif.).

**Verzhbinskiĭ, M. L. On the distribution of the roots of the *L*-transforms of entire transcendental functions.** Amer. Math. Soc. Translation no. 16, 54 pp. (1950).

Translated from Mat. Sbornik N.S. 22(64), 391–424 (1948); these Rev. 10, 443.

**Korevaar, J. Functions of exponential type bounded on sequences of points.** Ann. Soc. Polon. Math. 22 (1949), 207–234 (1950).

This is identical with part II of the author's thesis [Approximation and interpolation applied to entire functions, Thesis, University of Leiden, 1949; these Rev. 10, 694]. *I. M. Sheffer* (State College, Pa.).

***Siegel, Carl L. Analytic Functions of Several Complex Variables. Notes by P. T. Bateman.** Institute for Advanced Study, Princeton, N. J., 1950. ii+200 pp. $2.50.

Ces leçons sont essentiellement consacrées à une exposition d'ensemble de la théorie des fonctions automorphes de plusieurs variables complexes, théorie qui est due, pour une part, à Siegel lui-même. Voici le plan du livre. Les notions générales sur les fonctions analytiques de plusieurs variables, indispensables pour la suite, font l'objet des 3 premiers chapîtres. Les chapîtres IV à IX sont consacrés aux fonctions abéliennes et aux fonctions thêta. Au chapître X, on donne une théorie générale des fonctions automorphes attachées à un domaine borné $D$ et à un groupe discontinu $\Gamma$ d'automorphismes de $D$, tel que l'espace quotient $D/\Gamma$ soit compact. Quant aux 2 derniers chapîtres XI et XII, disons, en gros, qu'on y étudie des exemples de domaines bornés $D$ dont le groupe d'automorphismes $\Omega$ est transitif, et des sous-groupes discrets $\Gamma$ de $\Omega$, pour lesquels l'espace quotient $D/\Gamma$ n'est plus nécessairement compact.

Voici une analyse plus détaillée de l'ouvrage: Chap. I. Étude de la divisibilité dans l'anneau des fonctions holomorphes à l'origine (dans l'espace de $n$ variables complexes); cet anneau est à factorisation unique. Chap. II. On passe des propriétés de divisibilité locale (chap. I) aux propriétés de divisibilité globale dans un domaine: problèmes de Cousin. Les théorèmes de Cousin sont énoncés sans démonstration. Chap. III. Deux lemmes techniques. Chap. IV et V. On y montre essentiellement qu'une fonction abélienne (i.e., méromorphe de $n$ variables complexes, admettant $2n$ périodes indépendantes) est quotient de 2 fonctions de Jacobi (i.e., fonction holomorphe $h(z)$ telle que, pour chaque période $u$, $h(z+u)=h(z)e^{L_u(z)}$, où $L_u(z)$ est linéaire-complexe en $z$). Chap. VI et VII. Consacrés aux relations (de Frobenius) nécessaires et suffisantes entre les périodes d'une fonction abélienne, et aux théorèmes d'existence des fonctions de Jacobi par les séries thêta. Chap. VIII. Structure algébrique du corps des fonctions abéliennes dans le cas "non dégénéré" (où il existe une fonction abélienne qui ne soit pas fonction de $n-1$ variables au plus). Démonstration des 2 théorèmes fondamentaux: (1) il existe $n$ fonctions abéliennes analytiquement indépendantes ($n$ translatées d'une même fonction); (2) si $f_1, \cdots, f_n$ sont algébriquement indépendantes, toute autre $f$ abélienne satisfait à une équation $P(f, f_1, \cdots, f_n)=0$, où le degré en $f$ du polynome $P$ est borné. Dans le "cas dégénéré," le corps des fonctions abéliennes a la structure d'un corps de fonctions abéliennes, non dégénéré, à $r<n$ variables. Exemple d'un cas où $r=0$ (toutes les fonctions abéliennes sont constantes).

Chap. IX. Questions diverses en relation avec les fonctions abéliennes: problèmes (non résolus) concernant le corps des fonctions méromorphes sur une variété analytique (complexe) compacte; cas d'une variété algébrique, condition pour que son corps de fonctions soit isomorphe à un corps de fonctions abéliennes; démonstration topologique (de Lefschetz) des relations de Frobenius entre les périodes d'une fonction abélienne, etc. Chap. X. Cas d'un domaine borné $D$ et d'un groupe discontinu $\Gamma$ d'automorphismes de $D$: les séries de Poincaré; formes automorphes de poids $m\geqq 2$. Dans le reste du chapître, on suppose que l'espace quotient $D/\Gamma$ est compact; $D$ est alors nécessairement un domaine d'holomorphie. Théorèmes fondamentaux d'existence des fonctions automorphes: analogues aux 2 théorèmes fondamentaux du chapître VIII. Pour la démonstration du second, l'auteur utilise un théorème de H. Cartan sur les idéaux de fonctions holomorphes. Les 2 théorèmes fondamentaux entraînent l'existence d'un système de $n+2$ formes automorphes (de poids convenable) telle que toute fonction automorphe soit le quotient de 2 polynomes isobares par rapport à ces $n+2$ formes, lesquelles sont liées par une relation dont le premier membre est un polynome isobare [ce dernier point a aussi été prouvé par Hervé, par une autre méthode; voir C. R. Acad. Sci. Paris 226, 462–464 (1948); ces Rev. 9, 343].

Chap. XI. On considère désormais des domaines bornés $D$ dont le groupe d'automorphismes $\Omega$ est transitif; ce sont donc des espaces homogènes de groupes de Lie, et É. Cartan a déterminé tous ceux qui sont en outre "symétriques" (on ignore s'il y en a d'autres). L'auteur donne un exposé détaillé des 4 grandes classes de domaines bornés homogènes indiquées par É. Cartan, et en donne aussi des formes équivalentes, non bornées. Il démontre explicitement, dans le cas d'une de ces classes, que l'espace homogène est le quotient du groupe $\Omega$ par un sous-groupe compact maximal. Le chapître contient aussi une méthode pour construire des sous-groupes discontinus $\Gamma$ du groupe des automorphismes de la boule ouverte $B$, tels que $B/\Gamma$ soit compact. Chap. XII. Consacré au "groupe modulaire de degré $p$." Il opère dans le "demi-plan généralisé" et a été étudié par l'auteur [Amer. J. Math. 65, 1–86 (1943); ces Rev. 4, 242]. Le demi-plan généralisé est l'espace homogène quotient du groupe symplectique à $2p$ variables (réelles) par le groupe unitaire à $p$ variables (complexes); c'est un domaine de l'espace de $n=\frac{1}{2}p(p+1)$ variables complexes. Le "groupe modulaire" est le sous-groupe du groupe symplectique, défini par des matrices à coefficients entiers. Siegel donne une méthode de construction d'autres sous-groupes discrets $\Gamma$ du groupe symplectique $\Omega$ (sous-groupes qui sont donc discontinus dans le demi-plan généralisé $D$), tels que $D/\Gamma$ soit compact. Mais, si $\Gamma$ est le groupe modulaire, $D/\Gamma$ n'est pas compact. Les liens entre la théorie des fonctions abéliennes et celle des fonctions automorphes du groupe modulaire sont mis en évidence. Pour terminer, on donne la construction d'un domaine fondamental (non compact) du groupe modulaire, et une théorie abrégée des formes modulaires et des fonctions modulaires.

*H. Cartan* (Paris).

**Oka, Kiyoshi. Note sur les fonctions analytiques de plusieurs variables.** Kōdai Math. Sem. Rep., no. 5–6, 15–18 (1949).

Mitteilung über einige offene Probleme aus der Theorie der Funktionen mehrerer komplexer Veränderlichen (Zusammenhang zwischen Pseudo- und Regulärkonvexität von Bereichen und Ideale analytischer Funktionen).

*P. Thullen* (Panamá).

**Zygmund, A. A remark on functions of several complex variables.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 66–68 (1950).

L'auteur démontre le théorème suivant. Si $f(z_1, z_2, \cdots, z_k)$ est régulière dans l'hypersphère $S$: $\sum_{i=1}^{k}|z_i|^2<1$ et si l'intégrale $\int_{\sigma_r}\log^+|f(r_1e^{i\theta_1}, \cdots, r_ke^{i\theta_k})|\,d\sigma_r$ est bornée, $\sigma_r$ désignant la frontière de l'hypersphère $S_r$: $\sum_{i=1}^{k}r_i^2\leqq r^2$ et $d\sigma_r$ l'élément d'aire sur $\sigma_r$, alors en presque tout point de $\sigma_1$, la fonction $f$ a une limite "non tangentielle" finie. Ceci résulte d'un théorème de Calderón [Trans. Amer. Math. Soc. 68, 47–54 (1950); ces Rev. 11, 357] concernant les fonctions harmoniques dans $S$ et bornées angulairement en presque tout point de $\sigma_1$, et du fait que la fonction $\log^+ f$ est sousharmonique dans $S$. Rappelons que si $u$ est sousharmonique dans $S$ et satisfait à $\int_{\sigma_r}u^+d\sigma_r=O(1)$, cette fonction possède une limite "radiale" finie en presque tout point de $\sigma_1$ [Privaloff, Rec. Math. [Mat. Sbornik] N.S. 3(45), 3–25 (1938)]. *J. Lelong* (Lille).

**Hitotumatu, Sin. On integral formulas of analytic functions of several complex variables and some related problems.** Kōdai Math. Sem. Rep., no. 5–6, 11–14 (1949).

The author states the following theorem. If $D$ is a domain in the space of $n$ complex variables $z_1=x_1+iy_1, z_2, \cdots, z_n$, and if for $a<b$ its intersections with the half-spaces $x_1>a$, $x_1<b$ are domains of regularity, then so is $D$ itself. For $n=2$ this was previously given by K. Oka [Tôhoku Math. J. 49, 15–52 (1942); these Rev. 7, 290]. The details of the sketch of proof and of other remarks were not clear to the reviewer. *S. Bochner* (Princeton, N. J.).

**Behnke, Heinrich, und Sommer, Friedrich. Analytische Funktionen mehrerer komplexer Veränderlichen. Über die Voraussetzungen des Kontinuitätssatzes.** Math. Ann. 121, 356–378 (1950).

The authors prove the continuity theorem under several sets of very general hypotheses. The maximum principle is shown to be intimately related to the theorem. In fact, they show that the usual requirement of analyticity of the surfaces can, under suitable conditions, be replaced by the requirements that the surfaces be once continuously differentiable and that they fulfill the maximum principle. One of their theorems in this direction is as follows. Let $F_1, F_2, \cdots$ be $k$-dimensional, once continuously differentiable surfaces ($k<2n$) with boundaries $C_1, C_2, \cdots$. Let every regular function $f(z_1, \cdots, z_n)$ on $F_\mu$ assume its maximum on $C_\mu$. Let the surfaces $F_\mu$ converge uniformly toward a surface $F_0$ and let the boundaries $C_\mu$ converge uniformly toward the boundary $C_0$ of $F_0$. Let $g(z_1, \cdots, z_n)$ be a function which is regular and single-valued on $C_0$ and which remains single-valued after every possible continuation from $C_0$ onto $F_0$. If $g(z_1, \cdots, z_n)$ possesses at least one singularity on $F_0$, then there exists a $\mu_0$ such that $g$ possesses a singularity on each surface $F_\mu$ with $\mu>\mu_0$. The authors also study the relationship of the notion of pseudo-convexity to the continuity theorem. In particular, the latter relationship enables them to give criteria in analytical form for the validity of the continuity theorem on $(2n-1)$-dimensional hypersurfaces. In the final section, they study surfaces on which the continuity theorem does not hold.

*W. T. Martin* (Cambridge, Mass.).

**Behnke, Heinrich, und Stein, Karl. Elementarfunktionen auf Riemannschen Flächen als Hilfsmittel für die Funktionentheorie mehrerer Veränderlichen.** Canadian J. Math. 2, 152–165 (1950).

Cousin's first and second theorems concern the existence of analytic functions of several complex variables with prescribed poles and zeros respectively in a given domain $D$, and are generalizations to several variables of the theorems of Mittag-Leffler and Weierstrass. Previous formulations of the theorems have required that $D$ be schlicht. In the present paper, the authors prove the two theorems when $D$ is the direct product of $n$ open Riemann surfaces. For Cousin's second theorem, the components of $D$ must be simply connected with, at most, one exception. (The same restriction occurs in the case of Cousin's second theorem for schlicht domains.) The proofs of the two theorems use a generalized Cauchy integral representation for analytic functions of one complex variable on open Riemann surfaces which the authors have previously developed [Math. Ann. 120, 430–461 (1949); these Rev. 10, 696]. In the latter part of the paper, the authors generalize Behnke's general form of the continuity theorem to the case of meromorphic functions. In order to do this they first give a new proof of the continuity theorem for regular functions using the generalized Cauchy integral formula referred to above. They then carry over the proof to the case of meromorphic functions.

*W. T. Martin* (Cambridge, Mass.).

**Rothstein, Wolfgang. Die Fortsetzung vier- und höherdimensionaler analytischer Flächen des $R_{2n}$ ($n\geqq3$). (Cousinsche Verteilungen 2. Art).** Math. Ann. 121, 340–355 (1950).

Während einer zweidimensionalen analytischen Fläche des Raumes $R_4$ eine analytische Funktion einer komplexen Veränderlichen entspricht, werden vierdimensionale oder allgemein $(2n-2)$-dimensionale Flächen ($n\geqq2$) durch analytische Funktionen mehrerer komplexen Veränderlichen dargestellt, und daher sind ihre Singularitäten den einschränkenden Gesetzen unterworfen, welche für diese Funktionen bekannt sind, oder (was dasselbe bedeutet) gelten über die analytische Fortsetzbarkeit der mindestens vierdimensionalen analytischen Flächen analoge Sätze wie für Funktionen mehrerer komplexen Veränderlichen. Dies nachzuweisen, ist der Sinn obiger Arbeit. Die analytischen Mannigfaltigkeiten werden durch "algebraische Verteilungen" (Cousinsche Verteilungen 2. Art) gegeben. Die Aussagen werden meist für vierdimensionale Mannigfaltigkeiten des Raumes $R_8$ gemacht, gelten aber auch für $(2n-2)$-dimensionale Flächen und nach sinngemässen Abänderungen noch für $2k$-dimensionale des Raumes $R_{2n}$ ($k\geqq2$; $n\geqq3$).

Im Mittelpunkt steht auch hier der Kontinuitätssatz, dessen Gültigkeit für algebraische Verteilungen bewiesen wird. Aus den übrigen Aussagen seien einige herausgegriffen: Liegt im Innern eines schlichten beschränkten Regularitätsbereiches $\mathfrak{B}^4$ des $R_8$ eine wesentliche Singularität einer analytischen vierdimensionalen Fläche $\mathfrak{G}^4$, so muss $\mathfrak{G}^4$ auf dem Rande von $\mathfrak{B}^4$ singulär werden, falls $\mathfrak{G}^4$ diesen schneidet. In einer allgemeineren Form wird bewiesen: $\mathfrak{P}^4$, ganz im innern von $\mathfrak{B}^4$ liegend, sei ein pseudokonvexer Polyeder (der Regularitätsbereich $\mathfrak{B}^4$ ist durch solche vom Innern her approximierbar); ist dann $V$ eine im Restbereich $\mathfrak{B}^4-\mathfrak{P}^4$ algebraische Verteilung, so ist $V$ eindeutig ins Innere von $\mathfrak{B}^4$ fortsetzbar (Satz I*). Dieser Satz gilt nicht mehr für zweidimensionale Flächen des $R_4$. Schliesslich gilt: Ist die Fläche $\mathfrak{G}^4$ algebraisch in allen Punkten einer algebraischen

Fläche $\mathfrak{F}^4$ des abgeschlossenen Raumes $R_4$ und schneidet $\mathfrak{G}^4$ überhaupt $\mathfrak{F}^4$, so ist auch $\mathfrak{G}^4$ eine algebraische Fläche (Sätze II und II**). Es interessiert, dass die Sätze über die Fortsetzbarkeit von algebraischen Verteilungen nur für Regularitätsbereiche angegeben werden, während die analogen Aussagen über Funktionen mehrerer komplexen Veränderlichen für beliebige Bereiche gelten. Diese Beschränkung kann nur teilweise aufgehoben werden; sie ist z.B. beim oben zitierten Satze I* notwendig.

*P. Thullen* (Panamá).

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## *Theory of Series*

**Pólya, G. Remarks on power series.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 199–203 (1950).

The author states and proves three unrelated theorems. (1) Let $f(z)=e^{-cz}f_1(z)$ where $c\geqq 0$ and $f_1(z)$ is an entire function of genus zero having positive zeros only, the least one of which is $\gamma$. Let $-zf'(z)/f(z)=\sum_1^\infty s_n z^n$, $1/f(z)=\sum_0^\infty t_n z^n$. Then the ratios $s_n/s_{n+1}$ decrease and $t_n/t_{n+1}$ increase monotonically with $n$ to the limit $\gamma$. The first ratio is constant for $n\geqq 2$ if and only if $\gamma$ is the only zero of $f(z)$, for $n\geqq 1$ if also $c=0$. The second ratio is a constant if and only if $f(z)$ is a polynomial of degree one. (2) If $\sum_0^\infty a_n z^n$ is a convergent power series but not a polynomial, then there is a sequence $\{\epsilon_n\}$ with $\epsilon_n=-1$ or $+1$ such that $\sum_0^\infty \epsilon_n a_n z^n$ does not satisfy any algebraic differential equation. The proof is based upon a theorem of H. Grönwall [Öfversigt K. Vetenskaps-Akademiens Förhandlingar (Stockholm) **55**, 387–395 (1898)] according to which a gap series $\sum_1^\infty a_{n_k} z^{n_k}$ with $n_{k+1}/n_k>k$ cannot satisfy such an equation. (3) A power series which satisfies formally an algebraic equation is necessarily convergent and actually satisfies the equation. This is an extreme case of results previously announced by the author without proof [C. R. Acad. Sci. Paris **201**, 444–445 (1935)]. *E. Hille* (New Haven, Conn.).

**Wright, E. M. On the coefficients of power series having exponential singularities. II.** J. London Math. Soc. **24**, 304–309 (1949).

[For part I see the same J. **8**, 71–80 (1933).] The author finds the leading term in the asymptotic expansion for large $n$ of $c_n$, where

$$f(x)=(1-x)^{-\beta}g(x)e^{P(x)}=\sum_{n=0}^{\infty}c_n x^n,$$

$g(x)$ is any function regular for $|x|\leqq 1$, $g(1)\neq 0$ and $P(x)$ is a sum of terms of the form $a_m(1-x)^{-\rho m}$. He gives without proof the similar results for $\{-x^{-1}\log(1-x)\}^\gamma f(x)$ and $\{a-\log(1-x)\}^\gamma f(x)$. *J. G. van der Corput.*

**Wright, E. M., and Yates, Barbara G. The asymptotic expansion of a certain integral.** Quart. J. Math., Oxford Ser. (2) **1**, 41–53 (1950).

Let $2\pi i\psi(y)$ be defined by the integral

$$\int_C u^{-\beta}\exp\{yu+Au^{-\rho}(1+\sum_{l=1}^{L}A_l u^{\sigma_l})\}du,$$

where the contour $C$ starts from $-\infty$ on the real axis, encircles the origin once counter-clockwise, and returns to $-\infty$. Here $L$ is any nonnegative integer, $\sigma, \sigma_1, \cdots, \sigma_L$ are positive numbers such that $0<\sigma_1<\cdots<\sigma_L<1+\rho$; $\sigma_l\neq\rho$ $(l=1, \cdots, L)$. The authors study the asymptotic expansion of $\psi(y)$ for $|\arg y|<\frac{1}{2}\pi$ and large values of $|y|$. The case $L=0$, where $\psi(y)$ reduces to a generalized Bessel function, has been studied by E. M. Wright [Proc. London Math. Soc. (2) **38**, 257–270 (1934)] using the saddle-point method. With the same method the authors obtain for large positive values $y$ and $|\arg A|\leqq\pi-K'$ that $\psi(y)=H(Y_0)$, where

$$Y_0=y\rho^{-1}|A|^{-1}\exp(-i\arg A/(\rho+1)),$$

$$H(Y)=\frac{Y^{\beta-\frac{1}{2}}e^{\Omega(Y)}}{\{2\pi y(\rho+1)\}^{\frac{1}{2}}}\{1+O(y^{-K''})\};$$

the exponent $\Omega(Y)$ is expressed in the form of a convergent series in descending powers (not usually integral) of $y$. A similar result holds for arbitrary values of $\arg A$. If $y$ is complex, then an obvious transformation and deformation of the contour of the integral for $\psi(y)$ furnish an asymptotic expansion of $\psi(y)$ in the sector $|\arg y|<\frac{1}{2}\pi-K$. These results will be applied by Yates in the theory of linear difference equations, and have been already applied by Wright [see the preceding review] for the determination of the asymptotic behavior for large $n$ of the coefficient of $x^n$ in the power series of a function with one or more exponential singularities on its circle of convergence.

*J. G. van der Corput* (Stanford University, Calif.).

**Leja, F. Sur une classe de fonctions homogènes et les séries de Taylor des fonctions de deux variables.** Ann. Soc. Polon. Math. **22** (1949), 245–268 (1950).

In previous papers [e.g., Ann. Acad. Polonaise Sci. Tech. Varsovie **7** (1945); these Rev. **10**, 111] the author has associated with a point set $E$ of the Euclidean $E_4$ a domain $D(E)$, describable by an inequality $t(z_1, z_2; E)<1$, such that if a power series $\sum_{n=0}^{\infty}P_n(z_1, z_2)$ is convergent in $E$ it is (absolutely) convergent in the (larger) point set $D(E)$. The present paper derives some further properties of the combinatory function $t(z_1, z_2; E)$ involved. *S. Bochner.*

**Jacobsthal, Ernst. Über eine Formel von Frobenius.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 14, 51–55 (1950).

The formula in question [S.-B. Preuss. Akad. Wiss. **1910**, 809–847, in particular, p. 835] expresses $(-1)^n B_n/n!$ as a certain determinant of the $n$th order. In the present paper an equivalent formula is derived making use of the fact that $B_n/n!$ is the coefficient of $x^n$ in the expansion of $(\sum_1^\infty x^k/k)^{-n}$. *L. Carlitz* (Durham, N. C.).

**Hsu, L. C. On a generalized Kelvin's series.** Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 280–288 (1949).

The author gives a symbolic formula which allows him to express a sum

$$F(s)=\sum_{m_1,\cdots,m_k=1}^{\infty}(-1)^{\sum m_j}\prod_{j=1}^{k}f(m_j)\Big(\sum_{l=1}^{k}m_l\Big)^{-s}$$

formally in terms of $\varphi(s)=(2^{1-s}-1)\zeta(s)$, where $f(s)$ is a polynomial, namely

$$F(s)=\Big(\sum_{n=0}^{k}\Delta^n f(1)\cdot\Delta^{-n}\Big)^k\binom{X-1}{k-1}\varphi(s),$$

where $\Delta$ applies only to functions of $X$ and afterwards $X$ is interpreted as an operator applying to $\varphi(s)$ by

$$X^p\varphi(s)=\varphi(s-p).$$

He also investigates the convergence of the series for various ranges of the parameters. For example,

$$\sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\infty}(-1)^{i+j+k}\frac{(ijk)^2}{(i+j+k)^3}\right\}\right]=\frac{1-4\log 2}{1008}.$$

*R. P. Boas, Jr.* (Evanston, Ill.).

**Tornheim, Leonard. Harmonic double series.** Amer. J. Math. **72**, 303–314 (1950).

The series $\sum_{p,q=1}^{\infty}{}'p^{-r}q^{-s}(p+q)^{-t}$ is termed a harmonic double series and is denoted by $(r, s, t)$. It is finite if and only if $r+t>1$, $s+t>1$, $r+s+t>2$. For certain values of $r, s, t$ it can be expressed as a polynomial in the values $\zeta(n)$ $(n=2, 3, \cdots)$ of the Riemann zeta-function. The paper contains several identities between the numbers $(r, s, t)$ and $\zeta(n)$. *H. D. Kloosterman* (Leiden).

**Sunouchi, Gen-ichirô. Notes on Fourier analysis. XVIII. Absolute summability of series with constant terms.** Tôhoku Math. J. (2) **1**, 57–65 (1949).

[For part XVII cf. Yano, same vol., 50–56 (1949); these Rev. **11**, 348.] Let $y_n=\sum_{k=0}^{n}a_{n,k}x_k$ be a transformation of the series $\sum x_k$ to a sequence $\{y_n\}$. Then (1) necessary and sufficient conditions for $y_n$ to be of bounded variation whenever $\sum|x_k|<\infty$ are that

(i) $$|a_{n,k}|<M_n$$

and

(ii) $$\sum_{n=0}^{\infty}|a_{n,k}-a_{n+1,k}|<M,$$

where $M_n$ and $M$ are independent of $k$. The author obtains this theorem, but with condition (i) omitted; the condition is necessary and sufficient for $y_n$ to exist whenever $\sum|x_k|<\infty$. The theorem is substantially equivalent to one obtained independently by K. Knopp and G. G. Lorentz for a series-to-series transformation [Arch. Math. **2**, 10–16 (1949); these Rev. **11**, 346]. The author deduces F. M. Mears's analogous theorem for a sequence-to-sequence transformation, and gives some applications of Mears's theorem [Ann. of Math. (2) **38**, 594–601 (1937)]. He obtains a Mercerian theorem which he states has been proved in a paper to appear elsewhere; it is also included in a theorem of C. E. Walsh [J. London Math. Soc. **17**, 13–17 (1942); these Rev. **4**, 79].

Next he discusses absolute summability by the sequence-to-sequence transformation $y_n=\sum_1^n p_k s_k/\sum_1^n p_k$, where $p_n>0$, $P_n=\sum_1^n p_k\to\infty$, which he calls $|R, p_n|$ summability; it is, in fact, equivalent to summability $|R, P_n, 1|$. He proves (2) a necessary and sufficient condition for $|R, p_n|$ summability to be equivalent to absolute convergence is that $P_n/P_{n-1}>\alpha>1$, where $\alpha$ is independent of $n$. For ordinary summability it is familiar that the same condition is sufficient [see G. H. Hardy, Divergent Series, Oxford, 1949, p. 59; these Rev. **11**, 25; an examination of the proof given there shows that the condition is also necessary].

The author proves (3) a sufficient condition for summability $|R, q_n|$ to imply summability $|R, p_n|$ is that $p_{n+1}/P_n<q_{n+1}/Q_n$; an examination of his proof shows that a necessary and sufficient condition for the same property is that $p_n/P_n<Kq_n/Q_n$, where $K$ is independent of $n$. For ordinary summability a sufficient condition was obtained by Cesàro and rediscovered by Hardy [E. Cesàro, Atti Accad. Lincei Rend. (4) **4**$_1$, 452–457 (1888); G. H. Hardy, Quart. J. Pure Appl. Math. **38**, 269–288 (1907)]; a necessary and sufficient condition was given by H. L. Garabedian and W. C. Randels [Duke Math. J. **4**, 529–533 (1938)].

Finally the author discusses the function-to-function transformation $y(x)=\int_0^\infty\varphi(x, t)ds(t)$. He shows that (4) if $\varphi(x, t)$ and $\Lambda(x)$ satisfy suitable conditions, then the bounded variation of $y(x)$ is equivalent to the bounded variation of $s(t)$ if and only if $\{\Lambda(x)\}^{-1}\int_0^x\Lambda(t)ds(t)$ is of bounded variation. In particular $\varphi(x, t)=e^{-t/x}$, $\Lambda(t)=t$, $s(t)=\sum_1^{[t]}a_n$ gives a Tauberian theorem for absolute summability due to J. M. Hyslop [J. London Math. Soc. **12**, 176–180 (1937)].

*L. S. Bosanquet* (London).

**Wintner, Aurel. On absolute Lambert sums.** Proc. Edinburgh Math. Soc. (2) **8**, 128–132 (1949).

Hardy and Littlewood [Proc. London Math. Soc. (2) **19**, 21–29 (1920)] have given the inclusion $L\subset A$ for the Lambert and Abel methods of summability. Using a similar method, the author proves the relation $|L|\subset|A|$ for the corresponding absolute summabilities. *G. G. Lorentz.*

**Rajagopal, C. T. On a Tauberian theorem of G. Ricci.** Proc. Edinburgh Math. Soc. (2) **8**, 143–146 (1949).

The main result is the following. For a series $\sum a_\nu$ let $A(x)$ denote its partial sum defined by $\nu\leqq x$, $\sigma_k(x)$ its Riesz sum of order $k$ and $f(s)$ its Dirichlet sum $D(\lambda_n)$. Then, for any set of numbers $\tau\geqq 0$, $K\geqq 0$, $k\geqq 0$, the conditions $\lim_{\eta\to 0+}\liminf_{x\to\infty}\inf_{x<y<x(1+\eta)}[A(y)-A(x)]\geqq-\tau$, $\sigma_k(x)-\sigma_{k+1}(x)\geqq -K$ and $\lim_{s\to 0+}f(s)=S$ imply

$$\limsup_{x\to\infty}A(x)\leqq S+\tau.$$

*G. G. Lorentz* (Kingston, Ont.).

**Kritikos, N. Une propriété de la moyenne arithmétique.** Bull. Soc. Math. Grèce **24**, 111–118 (1949). (Greek. French summary)

Let $n\geqq 3$ and $2\leqq m\leqq n-1$. Let $\beta(0, 1), \beta(0, 2), \cdots, \beta(0, n)$ be $n$ real numbers. Let $\beta(1, 1), \beta(1, 2), \cdots, \beta(1, n)$ denote in any order the $n$ arithmetic means

$$[\beta(1, 1)+\beta(1, 2)+\cdots+\beta(1, m)]/m,$$
$$[\beta(1, 2)+\beta(1, 3)+\cdots+\beta(1, m+1)]/m, \cdots,$$
$$[\beta(1, n)+\beta(1, 1)+\cdots+\beta(1, m-1)]/m.$$

Let $\beta(2, 1), \cdots, \beta(2, n)$ be defined similarly in terms of $\beta(1, 1), \cdots, \beta(1, n)$ and so on. If $1\leqq j_k\leqq n$ for each $k$ then, as $k\to\infty$, $\beta(k, j_k)\to[\beta(0, 1)+\beta(0, 2)+\cdots+\beta(0, n)]/n$.

*R. P. Agnew* (Ithaca, N. Y.).

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## *Fourier Series and Generalizations, Integral Transforms*

**Zygmund, Antoni. Trigonometric Interpolation.** University of Chicago, 1950. iv+99 pp.

"The purpose of these notes is to present those aspects of trigonometric interpolation which resemble the theory of Fourier series." There are a number of novelties of presentation; several topics which are not new but are little known are discussed; and there are some new results. Attention is almost exclusively confined to interpolation at equidistant points; let $I_n(x; f)$ denote the trigonometric polynomial of order $n$ coinciding with a given $f(x)$ (of period $2\pi$) at $2n+1$ equidistant points. After introductory remarks, the author first discusses $I_n(x; f)$ as the Fourier series of $f$ with respect to the finite system $\frac{1}{2}$, $\cos x$, $\sin x$, $\cdots$, $\cos nx$, $\sin nx$, and

weight $d\omega_{2n+1}(x)$, where $\omega_{2n+1}$ is the appropriate step function; in particular, the $k$th partial sum $I_{n,k}(x;f)$ of $I_n(x;f)$ acquires significance. The case of an even number of fundamental points is then discussed and M. Riesz's interpolation formula is obtained and applied (Bernstein theorem and generalizations). The next chapter discusses the Fourier-Lagrange coefficients $c_\nu^{(n)}=(2\pi)^{-1}\int_0^{2\pi}f(t)e^{-i\nu t}d\omega_{2n+1}(t)$. Then the convergence of $I_n(x;f)$ is compared with the convergence of the partial sums $S_n(x;f)$ of the Fourier series. In particular, if $f(x)$ is continuous and its modulus of continuity is $o(1/\log\delta^{-1})$, then $I_{n,\nu}(x;f)\to f(x)$ uniformly $(n\geqq\nu)$ [cf. D. Jackson, Amer. Math. Monthly 34, 401–405 (1927)]. The next topic is the Jackson polynomials and other polynomials corresponding to modifications of the interpolation process, with better convergence properties than $I_n(x;f)$. Mean convergence and strong summability of $I_n(x;f)$ are then discussed. The next chapter is devoted to divergence; in particular, it is shown that there is a continuous $f$ for which $I_n$ diverges almost everywhere (and pointed out that divergence can even occur everywhere); for such an $f$ the Fourier series may converge uniformly. Other differences are shown by the facts that for continuous $f$ we have $I_n=o(\log n)$ and not necessarily more, while $S_n=o(\sqrt{\log n})$; and $I_{n_k}$ may diverge almost everywhere for a lacunary $\{n_k\}$. The last chapter contains new results on the polynomials $\tilde{I}_{n,k}$ conjugate to $I_{n,k}$. The role of the conjugate function $\tilde{f}$ in the theory of Fourier series is played by $\hat{f}(x)=\lim\hat{f}_n(x)$, where

$$\hat{f}_n(x)=-\pi^{-1}\int[f(t)-f(x)]\tfrac{1}{2}\cot\tfrac{1}{2}(t-x)d\omega_{2n+1}(t),$$

with integration over the part of $[0,2\pi]$ where

$$|t-x|\geqq 2\pi/(2n+1).$$

However, $\hat{f}(x)$ may exist nowhere for a continuous $f$; under a variety of stronger conditions, it does exist and equals $\tilde{f}$ and is the limit of $\tilde{I}_{n,k}$. *R. P. Boas, Jr.* (Evanston, Ill.).

**Natanson, I. P. On the degree of approximation to a continuous function of period $2\pi$ by means of its Poisson integral.** Doklady Akad. Nauk SSSR (N.S.) **72**, 11–14 (1950). (Russian)

Let $f(x)$ be a continuous function of period $2\pi$, let $\omega(\delta)$ be the modulus of continuity of $f$, and let $P(x,r)$ be the Poisson integral of $f$. Let $0<r_0<1$. Then for all $x$ and for $r_0\leqq r\leqq 1$,

$$|P(x,r)-f(x)|\leqq K\omega[(1-r)|\log(1-r)|]$$

where $K_0$ depends on $r_0$ only. Let $\mathrm{Lip}_1\alpha$ be the class of functions $f$ such that $|f(x)-f(y)|\leqq|x-y|^\alpha$, and let $u(\alpha,r)=\sup|P(x,r)-f(x)|$ for all $x$ and all $f\in\mathrm{Lip}_1\alpha$. Then for $r\to 1$ we have the asymptotic formulas

$$u(1,r)=2\pi^{-1}(1-r)\log(1-r)+O(1-r),$$
$$u(\alpha,r)=(1-r)^\alpha\sec\tfrac{1}{2}\pi\alpha+O(1-r),\qquad 0<\alpha<1.$$

*A. Zygmund* (Cambridge, Mass.).

**Timan, A. F., and Timan, M. F. Generalized modulus of continuity and best approximation in the mean.** Doklady Akad. Nauk SSSR (N.S.) **71**, 17–20 (1950). (Russian)

Let $f(x)$ be an integrable function of period $2\pi$, and let $\|f\|_p=(\int_0^{2\pi}|f|^p dx)^{1/p}$ for $1\leqq p\leqq\infty$. Let $E_{n,p}[f]$ be the best approximation of $f$, in metric $L^p$, by trigonometric polynomials of order not exceeding $n$, and let

$$\omega_p(f,h)=\sup_{|t|\leqq h}\|f(x+t)-f(x)\|_p$$

be the integral modulus of continuity of $f$ in $L^p$. Let $\Lambda_p$ be the class of functions $f$ such that

$$\|f(x+h)+f(x-h)-2f(x)\|_p=O(h).$$

It is known that $\Lambda_p$ coincides with the class of functions $f$ such that $E_{n,p}[f]=O(1/n)$ [see Zygmund, Duke Math. J. 12, 47–76 (1945); these Rev. 7, 60]. It is also known [loc. cit.] that for $f\in\Lambda_p$ we have $\omega_p(f,h)=O(h\log h)$, and that at least in the case $p=\infty$ this is the best possible estimate. The authors point out that this estimate can be improved for some other values of $p$. For example, $\omega_2(f,h)=O(h|\log h|^{\frac{1}{2}})$, if $f\in\Lambda_2$. For other values of $p$ their results are less complete. They state that if $f\sim\sum c_\nu e^{i\nu x}$, $1<p\leqq 2$, and if $(\sum_{|\nu|\geqq n}|c_\nu|^p)^{1/p}=O(1/n)$, then $\omega_q(f,h)=O(h|\log h|^{1/p})$, $q=p/(p-1)$. [It can be shown that if $f\in\Lambda_p$, $1\leqq p<\infty$, then $\omega_p(f,h)=O(h|\log h|^{1/p})$ for $1\leqq p\leqq 2$, and

$$\omega_p(h,f)=O(h|\log h|^{\frac{1}{2}})$$

for $p\geqq 2$; these results are best possible.] Let

$$\lambda_k^n=\cos(2k+1)\pi/2n,$$

$k=0,1,\cdots,n-1$, and let $U_n(f,x)=\sum_{k=0}^{n-1}\lambda_k^n S_k(x,f)$, where $S_k$ is the $k$th partial sum of the Fourier series of $f$. The authors state that for $f\in\Lambda_p$, a necessary and sufficient condition for $\omega_p(f,h)=O(h)$, $1<p<\infty$, is $U_n(f)=O(1)$.

*A. Zygmund* (Cambridge, Mass.).

**Rogosinski, W. W., and Szegö, G. Extremum problems for non-negative sine polynomials.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 112–124 (1950).

The authors consider sine polynomials

$$S(\theta)=b_1\sin\theta+\cdots+b_n\sin n\theta,$$

$b_1=1$, which are nonnegative in $(0,\pi)$. They determine the maximum of $S(\theta)$ for fixed $\theta$, the maxima of $b_2$, $b_3$, $b_{n-1}$, $b_n$, and show how to compute the maximum $b_k$ for other values of $k$. The results are too complicated to quote in detail here; we mention only three of the simplest:

$$\sum_{k=1}^{n}kb_k\leqq(n+1)(n+2)(n+3)/24$$

if $n$ is odd, $n(n+2)(n+4)/24$ if $n$ is even; $b_2\leqq 2\cos 2\pi/(n+3)$ if $n$ is odd, $2\cos\theta_0$ if $n$ is even, where $\theta_0$ is the smallest positive root of $(n+4)\sin\frac{1}{2}(n+2)\theta+(n+2)\sin\frac{1}{2}(n+4)\theta$; $-(n-1)/(n+3)\leqq b_n\leqq 1$ if $n$ is odd; $|b_n|\leqq n/(n+2)$ if $n$ is even. The method used is rather different from methods which have previously been used for trigonometric polynomials nonnegative over a whole period, and depends on the use of appropriate orthogonal polynomials.

*R. P. Boas, Jr.* (Evanston, Ill.).

**van der Waerden, B. L. Über Landau's Beweis des Primzahlsatzes.** Math. Z. **52**, 649–653 (1950).

Let $g(\varphi)=a_0+a_1\cos\varphi+\cdots+a_n\cos n\varphi\geqq 0$, with $a_k\geqq 0$ and $a_0<a_1$. Landau's problem was to choose $g(\varphi)$ to minimize $P_n=g(0)/(a_1-a_0)$. By a result of Tschakaloff [Ann. Scuola Norm. Super. Pisa (2) **9**, 13–26 (1940); these Rev. 1, 225, 400], $U=\inf P_n\leqq P_9<5.903$. The author shows that $U>5.864$. *R. P. Boas, Jr.* (Evanston, Ill.).

**Boas, R. P., Jr. Fourier series with a sequence of positive coefficients.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 35–37 (1950).

Suppose $f(x)$ has a Fourier series $\sum_{-\infty}^{\infty}c_n e^{inx}$ with real coefficients and suppose that the changes in sign in the

coefficients occur fairly sparsely (i.e., if a change in sign does not occur at $k_n$, then $|k_n-nB|<L$). If in a sufficiently large neighborhood of $x=0$, $f^{(\rho)}(x)$ exists and is integrable, then $f^{(\rho-1)}(x)$ exists everywhere. The result also holds if changes in sign in the coefficient sequence are ignored if they occur sufficiently close to 0. A corresponding result is given for the radial boundary function of a power series with real coefficients. *P. Civin* (Eugene, Ore.).

**Tsuchikara, Tamotu. A theorem on Riemann sum. Notes on Fourier analysis. XIII.** J. Math. Soc. Japan 1, 232–234 (1949).

Let $f(x)$ be a function which is integrable in $(0, 2\pi)$, has period $2\pi$, and has Fourier series

$$f(x)\sim a_0/2+\sum(a_n\cos nx+b_n\sin nx).$$

Define $f_n(x)$ as the $n$th Riemann sum of $f$, so that $f_n(x)=(1/n)\sum_{k=1}^{n}f(x+2\pi k/n)$. If

$$\lim_{n\to\infty}\sum_{\nu=1}^{\infty}(|a_{n\nu}-a_{n(\nu+1)}|+|b_{n\nu}-b_{n(\nu+1)}|)=0,$$

then for almost all $x$ there exists a sequence of integers $\{m_k\}$ (depending on $x$) such that $\lim f_{m_k}(x)=\int_0^{2\pi}f(x)dt$ and $\lim k/m_k=1$. The condition on the Fourier coefficients is satisfied in particular if $\sum(|a_n-a_{n+1}|+|b_n-b_{n+1}|)<\infty$, or if $a_n$, $b_n$ are nonincreasing. This result forms a complement to theorems of Marcinkiewicz and Zygmund [Fund. Math. 28, 131–166 (1936)] and Ursell [J. London Math. Soc. 12, 229–232 (1937)]. *R. M. Redheffer* (Cambridge, Mass.).

**Sz.-Nagy, Béla. Séries et intégrales de Fourier des fonctions monotones non bornées.** Acta Univ. Szeged. Sect. Sci. Math. 13, 118–135 (1949).

The author begins by considering the Fourier sine coefficients $b_n$ and cosine coefficients $a_n$ of a function $f(x)$ which decreases and is bounded below in $0<x<\pi$, with $xf(x)\in L$ for sine series, $f(x)\in L$ for cosine series. Let $0<\gamma\leqq 1$. Then $\sum n^{-\gamma}b_n$ converges absolutely if and only if $x^{\gamma-1}f(x)\in L$; $\sum n^{-\gamma}a_n$ converges if and only if $x^{\gamma-1}f(x)\in L$ for $\gamma<1$, $f(x)\log x\in L$ for $\gamma=1$ [the case $\gamma=1$ is due to Zygmund, Fund. Math. 13, 284–303 (1929); the author's proof is different]. More general sufficient conditions are given, and in particular a condition is given for the absolute convergence of the Fourier series of a function $F(x)$ whose graph is made up of a finite number of convex or concave arcs, namely the existence of $\int_0 u\log u\,|d[F'(\alpha+u)+F'(\alpha-u)]|$ for each point $\alpha$ separating two such arcs (this allows the graph to have vertical tangents at $\alpha$). Analogous results are given for Fourier integrals, where in each case there are two theorems corresponding respectively to integrability at 0 and $\infty$. *R. P. Boas, Jr.* (Evanston, Ill.).

**Sz.-Nagy, Béla. Méthodes de sommation des séries de Fourier. I.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 204–210 (1950).

A triangular matrix $\Lambda=(\lambda_{nk})$ ($n=0, 1, \cdots; k=0, 1, \cdots, n$; $\lambda_{n0}=1$) of real or complex numbers is said to be of type $F$ if, for every Fourier series

$$\sum_0^{\infty}A_k(x)=\tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)\sim f(x),$$

the linear means $\sigma_n(x)=\sigma_n(\Lambda, f; x)=\sum_{k=0}^{n}\lambda_{nk}A_k(x)$ converge to $f(x)$ at every Lebesgue point of $f$, and if the convergence is uniform over every closed interval of continuity of $f$. The following theorem partly generalizes earlier results of Hille and Tamarkin [Trans. Amer. Math. Soc. 34, 757–783 (1932)] and of Nikolskiĭ [Izvestiya Akad. Nauk SSSR. Ser. Math. 12, 259–278 (1948); these Rev. 10, 247]. A matrix $\Lambda$ is of type $F$ if $\lim_{n\to\infty}\lambda_{nk}=1$ ($k=1, 2, \cdots$) and

$$\sum_{k=0}^{n-1}\left(\sum_{i=n-k}^{n}(n-k)/i\right)|\Delta^2\lambda_{nk}|<\text{constant},$$

where $\Delta^2\lambda_{nk}=\lambda_{nk}-2\lambda_{n,k+1}+\lambda_{n,k+2}$ ($k=0, \cdots, n-1$; $\lambda_{n,n+1}=0$). *A. Zygmund* (Cambridge, Mass.).

**Izumi, Shin-ichi. Determination of function by its Fourier series. Notes on Fourier analysis. XII.** J. Math. Soc. Japan 1, 226–231 (1949).

[For note XI see the author and Kawata, Tôhoku Math. J. 47, 14–17 (1940); these Rev. 2, 94.] Suppose $f(x)$ is integrable and periodic. Let $\overline{S_n}(x)$ and $\overline{\sigma_n}(x)$ represent the partial sums and the Fejér means of the conjugate of the Fourier series of $f(x)$, and let $\psi_x(t)=f(x+t)-f(x-t)$. If there is an $h(x)$ such that (1) $\int_0^t\{\psi_x(u)-h(x)\}=o(t)$ and if (2) $\int_{\pi/n}^{\pi}|\psi_x(u)-\psi_x(u-\pi/2n)|u^{-1}du=o(\log n)$, then $\lim_{n\to\infty}\overline{S_n}(x)/\log n=-h(x)/\pi$. If (2*) represents (2) with $O$ instead of $o$, then (1) and (2*) yield

$$\lim_{n\to\infty}\overline{\sigma_{2n}}(x)-\overline{\sigma_n}(x)=\pi^{-1}(\log 2)h(x).$$

*P. Civin* (Eugene, Ore.).

**Izumi, Shin-ichi. Notes on Fourier analysis. XVI.** Tôhoku Math. J. (2) 1, 144–166 (1950).

The paper consists of four independent parts. In part 1 the conditions $\int_0^t\varphi(u)du=o(t)$ and

$$\int_{\pi/n}^{\pi}t^{-1-k}|\varphi(t)-\varphi(t-n^{-1}\pi)|dt=o(n^k),$$

where $\varphi(t)=f(x+t)+f(x-t)-2f(x)$, are shown to be sufficient for the Fourier series of $f(t)$ to be summable $(C, k)$ at $t=x$, for $k>-1$. In part 2 several examples are given to show that $\int_0^t|\varphi(u)|du=o(w(t))$ cannot be replaced by $\int_0^t\varphi(u)du=o(w(t))$ and $\int_0^t|\varphi(u)|du=O(w(t))$. In particular it is established that under the second set of conditions $S_n(x)\neq o(\log n)$ when $w(t)=1$, and that the Hardy-Littlewood convergence theorem does not hold (with the usual coefficient condition) when $w(t)=t/\log t^{-1}$. In part 3 the author considers the definition of the Stieltjes integral $\int_0^{2\pi}f(x)dg(x)$ in terms of the Fourier coefficients of $f(x)$ and $g(x)$ and gives a sufficient condition for the existence of the integral. The same procedure is adopted for the Hellinger integral $\int_{-\pi}^{\pi}[df(x)dg(x)]/dx$. Part 4 is devoted to obtaining analogous results for class Lip $(\alpha, r)$ to the results of Kac [Ann. of Math. (2) 44, 411–415 (1943); these Rev. 5, 4] for class Lip $\alpha$. *P. Civin* (Eugene, Ore.).

**Matsuyama, Noboru. Notes on Fourier analysis. XIV. Absolute Cesàro summability of Fourier series.** Tôhoku Math. J. (2) 1, 40–45 (1949).

[The paper was incorrectly numbered "X" in the original.] It is known that, if $0<\alpha\leqq 1$, $\alpha p>1$ and $f(x)\in\text{Lip }(\alpha, p)$, i.e.,

$$\left(\int_{-\pi}^{\pi}|f(x+h)-f(x)|^p dx\right)^{1/p}=O(h^{\alpha}),$$

then for all $x$, (i) if $1<p\leqq 2$ the Fourier series of $f(x)$ is summable $|C, p^{-1}-\alpha+\epsilon|$, $\epsilon>0$, (ii) if $p\geqq 2$ the Fourier series is summable $|C, \frac{1}{2}-\alpha+\epsilon|$. When $\alpha p\leqq 1$, if a further local condition is satisfied, then for the corresponding value of $x$, (i) and (ii) also hold [H. C. Chow, Proc. London Math.

Soc. (2) **43**, 484–489 (1937); J. London Math. Soc. **17**, 17–23 (1942); these Rev. **4**, 38].

Here the author gives the result: if $0<\alpha\leqq 1$, $1\leqq p\leqq 2$, $\alpha p\leqq 1$ and $f(x)\varepsilon \mathrm{Lip}(\alpha, p)$, then the Fourier series of $f(x)$ is summable $|C, p^{-1}-\alpha+\epsilon|$ for almost all $x$. The proof contains many misprints and is difficult to follow without reference to a paper of G. H. Hardy and J. E. Littlewood [Math. Z. **28**, 612–634 (1928)]. He obtains a second result: that when $\alpha p>1$, the Fourier series is summable $|C, 1-p^{-1}|$. But if $1<p\leqq 2$, $1-p^{-1}>0>p^{-1}-\alpha$; and if $p>2$, $1-p^{-1}>\frac{1}{2}>\frac{1}{2}-\alpha$. Thus in both cases this result is inferior to the corresponding one of Chow.

*L. S. Bosanquet* (London).

**Sunouchi, Gen-ichirô. Notes on Fourier analysis. XI. On the absolute summability of Fourier series.** J. Math. Soc. Japan **1**, 122–129 (1949).

The author begins by proving that if $\sum|u_n|$ converges then $\sum n^{-\gamma}u_n$, $0<\gamma<1$, is summable $(C, -\gamma)$. This is included in a general theorem obtained by E. Kogbetliantz, but not proved in this case [Bull. Sci. Math. (2) **49**, 234–256 (1925); see also Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques, Mémor. Sci. Math., no. 51, Gauthier-Villars, Paris, 1931, p. 28, where this case is stated]. He then observes that, in the light of this result, certain theorems of H. C. Chow (partly obtained also by K. K. Chen) concerning the absolute Cesàro summability of a Fourier series for negative orders [Chow, J. London Math. Soc. **17**, 17–23 (1942); Chen, Amer. J. Math. **66**, 299–312 (1944); these Rev. **4**, 38; **5**, 262] are immediate corollaries of an example in A. Zygmund's Trigonometrical Series [Warszawa-Lwów, 1935, p. 143, example 6] and hence are corollaries of known results due to S. Bernstein and others concerning sums of powers of the coefficients of a Fourier series [for reference, see Zygmund, op. cit.].

The author next proves two theorems on $|C, 1|$-summability factors in a Fourier series. These were obtained by H. C. Chow [J. London Math. Soc. **16**, 215–220 (1941); these Rev. **4**, 37].

He then proves that $|C, 1|$ summability is a local property for Fourier series of functions of $L^p$ $(p>1)$. This is known to be false for functions of $L$ [L. S. Bosanquet and H. Kestelman, Proc. London Math. Soc. (2) **45**, 88–97 (1939); see also R. Mohanty, J. London Math. Soc. **25**, 67–72 (1950); these Rev. **11**, 592]. He obtains a local sufficient condition:

$$\left(t^{-1}\int_0^t |f(x+u)+f(x-u)|^s du\right)^{1/s}=o(\log^{-1-\epsilon}(t^{-1})),$$

$$s>1,\ \epsilon>0,$$

by quoting a result of T. Tsuchikura [see the following review]. In the case $p=2$ he adapts an example of A. Zygmund [Trans. Amer. Math. Soc. **55**, 170–204 (1944); these Rev. **5**, 230] to show that the Fourier series may be summable $|C, 1|$ in $(a, b)$, but nonsummable $|A|$ almost everywhere else in $(0, 2\pi)$. *L. S. Bosanquet* (London).

**Tsuchikura, Tamotsu. Notes on Fourier analysis. III. Convergence character of Fourier series at a point.** Math. Japonicae **1**, 135–139 (1949).

Let

$$M_p(|\varphi(t)|)=\left(t^{-1}\int_0^t |\varphi(u)|^p du\right)^{1/p},$$

$$M_k(|s_n|)=\left(n^{-1}\sum_0^n |s_\nu|^k\right)^{1/k},$$

where $\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$ and $s_n$ is the $n$th partial sum of the Fourier series of $f(t)$ at $t=x$. It is known that: (1) if $p>1$, $f\varepsilon L(0, 2\pi)$ and $M_p(|\varphi(t)|)=o(1)$, then $M_k(|s_n|)=o(1)$ for $k>0$; (2) if $p>1$ and $|\varphi(t)|^p/t\varepsilon L(0, \pi)$, then $\sum|s_n|^k/n<\infty$ for $k\geqq p$ [G. H. Hardy and J. E. Littlewood, Proc. London Math. Soc. (2) **26**, 273–286 (1927); Duke Math. J. **2**, 354–382 (1936)]. In particular, the hypothesis in (2) is satisfied if $M_p(|\varphi(t)|)=O(\log^{-1}(2\pi t^{-1}))$ provided that $\epsilon>p^{-1}$.

Here the author proves that: (3) if $p>1$, $\epsilon>0$, and $M_p(|\varphi(t)|)=O(\log^{-\epsilon}(t^{-1}))$ then (i) $M_k(|s_n|)=O(\log^{-\epsilon} n)$ for $k>0$, (ii) $\sum|s_n|^k/n<\infty$ for $k>\epsilon^{-1}$, (iii) $\sum|s_n|^\rho/n\log n<\infty$ for $\rho>0$. The result (ii) is a corollary of (i), and (iii) is a corollary of (ii). If $k\geqq p>\epsilon^{-1}$, (ii) is included in (2), but not if $p>k$ or $p\leqq\epsilon^{-1}$. The author gives an example to show that (ii) is false with $k=\epsilon^{-1}$. There are similar results with $s_n-\sigma_n$ in place of $s_n$, where $\sigma_n$ is the $n$th arithmetic mean of the partial sums. *L. S. Bosanquet* (London).

**Izumi, Shin-ichi, and Matsuyama, Noboru. Notes on Fourier analysis. IV. On the absolute Riesz summability of Fourier series.** Math. Japonicae **1**, 140–150 (1949).

Let $\psi_0(t)=\varphi(t)$ and let

$$\psi_s(t)=\frac{1}{\Gamma(s)}\frac{1}{(\log(t^{-1}))^s}\int_t^\pi (\log(u/t))^{s-1}u^{-1}\varphi(u)du \qquad (s>0)$$

denote the logarithmic mean of order $s$ $(s\geqq 0)$ of

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}.$$

The authors give the following results. (1) If $f\varepsilon L$ and $\psi_s(t)=o(\log^{-\epsilon}(\epsilon^{-1}))$, then the Fourier series of $f(x)$ is summable $|R, \log n, s+1|$, where $s>0$, $\epsilon>0$. (2) If $f\varepsilon L$ and $|\psi_s(t)|^p/t\varepsilon L(0, \eta)$, then the Fourier series of $f(x)$ is summable $|R, \log n, s+1+\delta|$, where $p>1$, $\eta>0$, $s>0$, $\delta>0$. They point out that (1) was incorrectly asserted for $s=0$ by Izumi and Kawata [Tôhoku Math. J. (1) **45**, 134–144 (1938)], but that the result holds in this case if $f\varepsilon L^p$, $p>1$, by a theorem of T. Tsuchikura [see the preceding review, (3) (iii)]. The proof for $s>0$ depends on a lemma which is not proved; it may be obtained, for example, by modifying another theorem of Tsuchikura [preceding review, (3) (i), with $-\epsilon$ replaced by $s-\epsilon>0$, in which case the result is a local property of the function]. The authors state that (2) holds with $s=0$, $\delta=0$, provided $f\varepsilon L^p$, $p>1$; but this is not substantiated, since it is based on an incorrect proof of an earlier assertion [Izumi and Kawata, loc. cit., theorem 3], where there is confusion between the partial sums and modified partial sums of a Fourier series. Their proof of (2) (given only for $s\geqq 1$) shows that the hypothesis may be replaced by $|\psi_s(t)|/(t\log(2\pi t^{-1}))\varepsilon L$. It is plausible to expect that this condition can be replaced by the more general one: $\psi_s(t)=o(1)$ $|R, \log n, 1|$ or, what must be equivalent, $\varphi(t)=o(1)$ $|R, \log n, s+1|$.

The authors add the following result on strong summability. (3) If $f\varepsilon L$ and $\varphi(t)=o(\log^{-\epsilon}(t^{-1}))$, then (i) for $0<\epsilon<\frac{1}{2}$, $(n^{-1}\sum_0^n|s_\nu|^2)^{\frac{1}{2}}=o(\log^{\frac{1}{2}-\epsilon} n)$, (ii) for $\epsilon>\frac{1}{2}$,

$$\sum|s_n|^2/(n\log n)<\infty.$$

As it stands, (3) (i) is included in a theorem of Hardy and Littlewood; in fact even if $o(\log^{-\epsilon}(t^{-1}))$ is replaced by $o(1)$ in the hypotheses, the conclusion holds with $o(1)$ in place of $o(\log^{\frac{1}{2}-\epsilon} n)$ [G. H. Hardy and J. E. Littlewood, C. R. Acad. Sci. Paris **156**, 1307–1309 (1913); see A. Zygmund, Trigonometrical Series, Warszawa-Lwow, 1935, p. 240]. On the other hand, analogy with a later theorem of Hardy

and Littlewood suggests that the right version should contain the hypothesis $t^{-1}\int_0^t|\varphi(u)|\,du=o(\log^{-\epsilon}(t^{-1}))$ [see Hardy and Littlewood, Fund. Math. 25, 162–189 (1935)]. There are various obscurities of detail. *L. S. Bosanquet.*

**Matsuyama, Noboru. Notes on Fourier analysis. XX. On the Riesz logarithmic summability of the derived Fourier series.** Tôhoku Math. J. (2) 1, 91–94 (1949).

It is known that (A) if

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}\to\lim\ (C,\alpha)$$

[or $(R,\log n,\alpha)$] as $t\to 0$, then the Fourier series of $f(x)$ is summable $(C,\alpha+\delta)$ [or $(R,\log n,\alpha+\delta)$], $\delta>0$; (B) if the Fourier series of $f(x)$ is summable $(C,\alpha)$ [or $(R,\log n,\alpha)$], then $\varphi(t)\to\lim\ (C,\alpha+1+\delta)$ [or $(R,\log n,\alpha+1+\delta)$] [Bosanquet, Proc. London Math. Soc. (2) 31, 144–164 (1930); F. T. Wang, Tôhoku Math. J. (1) 40, 274–292 (1935)]. Theorems about the Cesàro summability of the derived Fourier series of $f(x)$ are obtained as corollaries of: (C) the derived Fourier series of $f(x)$ is summable $(C,\alpha+1)$ if and only if the Fourier series of $\psi(t)/t=\frac{1}{2}\{f(x+t)-f(x-t)\}/t$ at $t=0$ is summable $(C,\alpha)$, $\psi(t)/t$ being integrable in a suitable sense [A. Zygmund, Studia Math. 3, 77–91 (1931) (with $f$ and $\psi(t)/t\in L$); Bosanquet, Quart. J. Math., Oxford Ser. (1) 10, 67–74 (1939) (with $f\in L$ and $\psi(t)/t\in CL$ (Cesàro-Lebesgue), the last condition being necessary)].

Here the author proves: (1) if $\psi(t)/t\to\lim\ (R,\log n,\alpha)$, $\alpha>0$, then the derived Fourier series of $f(x)$ is summable $(R,\log n,\alpha+1+\delta)$; (2) if the derived Fourier series of $f(x)$ is summable $(R,\log n,\alpha+1)$, $\alpha>0$, then

$$\psi(t)/t\to\lim\ (R,\log n,\alpha+2+\delta).$$

He deduces (1) from (A) by means of the identity:

$$(*)\qquad D_{\alpha+1}(\omega)=R_{\alpha+1}(\omega)-(\alpha+1)R_\alpha(\omega)/\log\omega,$$

where $D_\alpha(\omega)$ and $R_\alpha(\omega)$ are the $(R,\log n,\alpha)$ means of the derived Fourier series of $f(x)$ and the Fourier series of $\psi(t)/t$ at $t=0$, respectively. But it seems to the reviewer that the hypothesis in (1) implies both (i) $R_\alpha(\omega)=o(\log\omega)$ and (ii) $R_{\alpha+1}(\omega)\to\lim$, and so $D_{\alpha+1}(\omega)\to\lim$. Thus (1) appears to hold with $\delta=0$. In proving (2) the author again appeals to (*). The argument is obscured by his stating two lemmas incorrectly and without proof or adequate reference. The reviewer observes that (*) may be solved in the form

$$(**)\qquad R_{\alpha+1}(\omega)=\frac{\omega}{(\log\omega)^{\alpha+1}}\int_\omega^\infty D_{\alpha+1}(u)u^{-2}(\log u)^{\alpha+1}du,$$

provided this integral converges, so that we have the Mercerian result that: if $D_{\alpha+1}(\omega)\to\lim$, then (i) $R_{\alpha+1}(\omega)\to\lim$ and (ii) $R_\alpha(\omega)=o(\log\omega)$; (2) is a consequence of (i) alone, by (B). The appropriate result corresponding to (C) now seems to be: (D) the derived Fourier series of $f(x)$ is $(R,\log n,\alpha+1)$ if and only if the Fourier series of $\psi(t)/t$ at $t=0$ is both summable $(R,\log n,\alpha+1)$ and has its partial sums $o(\log\omega)$ $(R,\log n,\alpha)$, $\psi(t)/t$ being integrable in a suitable sense. *L. S. Bosanquet* (London).

**Matsuyama, Noboru. Notes on Fourier analysis. X. On the summability of Fourier series.** J. Math. Soc. Japan 1, 116–121 (1949).

Let $I^\alpha$ and $L^\alpha$ denote operations of fractional integration and logarithmic integration of order $\alpha$, with bases 0 and 1 respectively. Let $A(\omega)=\sum_{n<\omega}a_n$, where $\sum a_n\cos nt$ is the Fourier series of $\varphi(t)$. Thus, for example, $L^\alpha A(\omega)/(\log\omega)^\alpha$ is the $(R,\log n,\alpha)$ mean for the series $\sum a_n$. It is convenient to state the following theorem. (A) If

$$I|L^{k-1}\varphi(t)|=O(t\log^{k+p}(t^{-1}))$$

as $t\to 0$, then a necessary and sufficient condition that $I^\alpha L^k A(\omega)=o(\omega^\alpha\log^k\omega)$ as $\omega\to\infty$ is that

$$I^\beta L^k\varphi(t)=o(t^\beta\log^k(t^{-1}))$$

as $t\to 0$ for some $\beta\geqq 0$, where $k\geqq 1$, $p\geqq 0$, $\alpha\geqq 0$. The case $k=1$, $p\geqq 0$ was given in effect by A. C. Offord and the reviewer in extension of a theorem of G. H. Hardy [Hardy, Quart. J. Math., Oxford Ser. (1) 2, 107–112 (1931); Bosanquet and Offord, Compositio Math. 1, 180–187 (1934), where it was possible to take $\alpha>-1$, since discrete means were used instead of continuous ones; a substantially equivalent result was given independently by F. T. Wang, Tôhoku Math. J. (1) 40, 142–159, 274–292 (1935)].

Here the author gives the case $k\geqq 1$, $p=0$ [the modulus signs are omitted in the enunciation, which is stated incorrectly with $\alpha>-1$]. The method of proof is to reduce the theorem by means of identities to the case $k=1$, $p=k$, of (A), which case may be established by an argument analogous to that of Offord and the reviewer. The author's treatment of the last stage (necessity) is inaccurate, and the references are also given incorrectly. *L. S. Bosanquet.*

**Wang, Shou-Jen. Summability factors of the conjugate series of a Fourier series at a given point.** Acad. Sinica Science Record 2, 245–249 (1949).

If $\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$ and $\varphi_\alpha(t)$ is the Cesàro mean of order $\alpha$ of $\varphi(t)$, it is known that if $\varphi_\alpha(t)$ is of bounded variation, then (for $\alpha\geqq 0$) the Fourier series $\sum A_n(x)$ of $f(x)$ is summable $|C,\alpha+\delta|$, $\delta>0$, and (in the case $0\leqq\alpha\leqq 1$) $\sum A_n(x)/\log^{1+\epsilon}n$, $\epsilon>0$, is summable $|C,\alpha|$ [Bosanquet, Proc. London Math. Soc. (2) 41, 517–528 (1936); Cheng, Duke Math. J. 15, 29–36 (1948); these Rev. 9, 580, 735]. If $\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$ and $\Psi_\alpha(t)$ is the Riemann-Liouville integral of order $\alpha$ of $\psi(t)$, it is known that if (*) $\Psi_\alpha(+0)=0$ and (**) $\int_{0+}^\pi t^{-\alpha}|d\Psi_\alpha(t)|<\infty$, then (for $\alpha\geqq 0$) the allied series $\sum B_n(x)$ of $f(x)$ is summable $|C,\alpha+\delta|$ [Bosanquet and Hyslop, Math. Z. 42, 489–512 (1937)]. Here the author proves (for $0\leqq\alpha\leqq 1$) that if (*) and (**) hold, then $\sum B_n(x)/\log^{1+\epsilon}n$ is summable $|C,\alpha|$. The result is false with $\alpha=0$, $\epsilon=0$. *L. S. Bosanquet* (London).

**Szász, O. On the Gibbs' phenomenon for Euler means.** Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 107–111 (1950).

Suppose $f(t)\sim\sum_{n=1}^\infty b\sin nt$ with

$$\lim_{\lambda\to 1}\limsup_{n\to\infty}\sum_{\nu=n}^{\lambda n}(|b_\nu|-b_\nu)=0.$$

If $f(t)$ has a simple jump at $t=0$, the Euler means of order $r$, $0<r\leqq 1$, $\sigma_n(t,r)=\sum_{\nu=0}^n\binom{n}{\nu}r^\nu(1-r)^{n-\nu}s_\nu(t)$, exhibit a Gibbs phenomenon somewhat akin to that of the partial sums $s_\nu(t)$ of the Fourier series of $f(t)$. In particular if $nt_n\to T$ and $nt_n^2\to 0$ as $n\to\infty$, then

$$\lim_{n\to\infty}\sigma_n(t_n,r)=(2/\pi)f(0+)\int_0^{rT}y^{-1}\sin y\,dy$$

and for any sequence $t_n\to 0$,

$$\limsup_{n\to\infty}\sigma_n(t_n,r)\leqq(2/\pi)f(0+)\int_0^{r}y^{-1}\sin y\,dy.$$

*P. Civin* (Eugene, Ore.).

**Sinvhal, S. D. On Cesaro non-summability of Fourier series.** Proc. Benares Math. Soc. (N.S.) **8**, no. 2, 23–29 (1946).

**Sinvhal, S. D. On Cesaro non-summability of Fourier series. II.** Proc. Benares Math. Soc. (N.S.) **9**, 43–47 (1947).

Let $f(x)$ be an $L$-integrable function of period $2\pi$. For $t>0$ let $\phi(t)+\phi_x(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$,

$$\lambda(t)=\int_0^t u^{-1}\phi(u)du.$$

It is shown in an example that the condition

$$\lambda(t)=o\{t|\log t|^{-\alpha}\},$$

$0<\alpha<\frac{1}{2}$, need not imply the summability $(C, 1)$ of the Fourier series of $f$ at the point $x$. *A. Zygmund.*

**Bochner, S., and Chandrasekharan, K. Lattice points and Fourier expansions.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 1–15 (1950).

This is a sequel to an earlier paper by the same authors [Quart. J. Math., Oxford Ser. (1) **19**, 238–248 (1948); these Rev. **10**, 431]. Using new theorems on the spherical summability of formally differentiated Fourier series of functions of several variables, the authors shorten the proofs of their previous theorems on the Cesàro summability of the developments, initiated by Voronoi and Hardy and associated with the number of lattice points in $k$-dimensional spheres. *A. Zygmund* (Cambridge, Mass.).

**Cheng, Min-Teh. The Gibbs phenomenon and Bochner's summation method. I.** Duke Math. J. **17**, 83–90 (1950).

Let $\{f_n(x)\}$ be a sequence of real-valued functions defined in a neighbourhood of a point $x_0$. The Gibbs set of $\{f_n(x)\}$ at $x_0$ is defined as the set of values $\eta$ for which $f_n(x)\to\eta$ when $n\to\infty$ and $x$ tends to $x_0$ through an appropriate sequence. The author considers the case where $f_n(x)$ is the $(R, n^2, \alpha)$ mean of the Fourier series of an integrable function $f(x)$. The latter is defined by

$$(*)\qquad S_R^\alpha(x)=\sum_{n^2<R}(1-n^2/R)^\alpha(a_n\cos nx+b_n\sin nx),$$

where $a_n$ and $b_n$ are Fourier coefficients of $f(x)$, and $\alpha>0$. The author proves that if $f(x)$ has a simple discontinuity at $x=x_0$, then the Gibbs set of $(*)$ at $x_0$ is a closed interval with center $\frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$ and with length $|2dP/\pi|$, where $d=f(x_0+0)-f(x_0-0)$,

$$P=(\tfrac{1}{2}\pi)^{\frac{1}{2}}2^\alpha\Gamma(\alpha+1)\int_0^\gamma J_\beta(t)t^{-\beta}dt\qquad(>\tfrac{1}{2}\pi),$$

$\beta=\frac{1}{2}+\alpha$, $\gamma$ being the smallest positive zero of the Bessel function $J_\beta(t)$. This theorem improves Kuttner's theorem [J. London Math. Soc. **19**, 153–161 (1944); these Rev. **7**, 154]. *S. Izumi* (Sendai).

**Gál, István Sándor. Sur la convergence d'interpolations linéaires. I. Fonctions bornées.** C. R. Acad. Sci. Paris **230**, 1374–1376 (1950).

The author considers generalized "interpolation formulas"

$$A_n(f, z)=\sum_{k=1}^{m_n}f(z_k^{(n)})\omega_k^{(n)}(z),\qquad n=1, 2, \cdots,$$

where the "fundamental points" $z_k^{(n)}$ $(k=1, 2, \cdots, m_n)$ belong to a closed set $K$ in the complex plane, and the "fundamental functions" $\omega_k^{(n)}(z)$, not necessarily polynomials, are regular and single-valued in $K$. He states without proof a number of necessary and sufficient conditions for the relation $(*)$ $A_n(f, z)\to f(z)$. The following result may serve as an example. A necessary and sufficient condition that $(*)$ hold at every point $z$ of continuity of every function $f(z)$ bounded in $K$ is

$$\text{(a)}\ \sum_{k=1}^{m_n}\omega_k^{(n)}(z)\to 1,\qquad \text{(b)}\ \sum_{|z^n-z|>\mu}|\omega_k^{(n)}(z)|\to 0\qquad(n\to\infty)$$

for every $\mu>0$, and

$$\text{(c)}\qquad \sum_{z^n\neq z}|\omega_k^{(n)}(z)|\leqq H(z)$$

with $H(z)$ independent of $z$. *A. Zygmund.*

**Alexits, Georges. Sur la convergence et la sommabilité presque partout des séries de polynomes orthogonaux.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 223–225 (1950).

Given is a nonnegative weight-function $w(x)$ in $\langle -1, 1\rangle$, and the $p_n(x)$ are the associated orthonormal polynomials. Let $(*)$ $f(x)\sim\sum_0^\infty c_n p_n(x)$ be the Fourier series of the integrable function $f(x)$ with respect to the $p_n(x)$, and let $f(\cos\vartheta)\sim\frac{1}{2}a_0+\sum_1^\infty a_k\cos k\vartheta$. Under the assumption that $0\leqq w(x)\leqq W(1-x^2)^{-\frac{1}{2}}$ the following theorems are proved. (i) If $\sum a_n^2\log^2 n<\infty$, then $(*)$ is convergent almost everywhere; if $\sum a_n^2(\log\log n)^2<\infty$, then $(*)$ is summable $(C, 1)$ almost everywhere. (ii) If the $p_n(x)$ are uniformly bounded, and if $\sum a_n^2\log n<\infty$, then $(*)$ converges almost everywhere. The proofs are based on the inequality

$$\sum_{k=1}^{\infty}\lambda_k c_k^2\leqq\tfrac{1}{2}W\pi\sum_{k=1}^{\infty}\lambda_k a_k^2,$$

where the $\lambda_k$ are nonnegative and nondecreasing. *W. W. Rogosinski* (Newcastle-upon-Tyne).

**Doss, Raouf. Some theorems on almost periodic functions.** Amer. J. Math. **72**, 81–92 (1950).

The author gives the following representation theorems for two classes of linear (i.e., additive and continuous) functionals. (I) Any linear functional $U(f)$ defined in the space of all Besicovitch almost periodic functions (or in a subspace with a fixed basis) is of the form (1) $U(f)=\mathfrak{M}\{f(x)\alpha(x)\}$ for some bounded Besicovitch almost periodic function $\alpha(x)$. (II) Any linear functional $U(f)$ defined in the space $S$ of Bohr almost periodic functions of basis $\{\beta_i\}$ is of the form (1), where $\alpha(x)$ is a summable function of finite upper mean whose Fourier coefficients exist for every exponent corresponding to the basis $\{\beta_i\}$. Conversely, for every such $\alpha(x)$, $U(f)$ defined by (1) exists and is linear in $S$. The author also gives necessary and sufficient conditions that a series $\sum a_n\exp(iu_nx)$ be the expansion of a Besicovitch almost periodic function. The condition is expressed in terms of Bochner sums. *R. H. Cameron* (Minneapolis, Minn.).

**Følner, Erling. A theorem on almost periodic functions of infinitely many variables.** Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 14, 15 pp. (1950).

The author defines a function $f(x)\equiv f(x_1, x_2, \cdots)$ to be fully periodic if its module of period vectors is not contained in any proper vector subspace of the space $R^\infty$ of all infinite-dimensional vectors $x$: $(x_1, x_2, \cdots)$. Here a vector subspace means a linear subset that is closed in the topology in which $x^n\to x$ means that $x_1^n\to x_1$, $x_2^n\to x_2$, $\cdots$. The author proves that an almost periodic function $f(x)$ is fully periodic if and only if the set of translated functions $f(x+h)$ is closed with respect to uniform convergence. *R. H. Cameron.*

**Levin, B. On functions determined by their values on a certain interval.** Doklady Akad. Nauk SSSR (N.S.) **70**, 757–760 (1950). (Russian)

Désignons par $T_\Omega$ la classe des fonctions presque-périodiques de Stepanoff ($S_p$) dont les exposants de Fourier appartiennent à l'ensemble $\Omega$ et désignons par $n_\Omega(t)$ le nombre de points de $\Omega$ situés dans $(-t, t)$. Posons

$$N_\Omega(R)=\int_0^R [n(t)/t]dt.$$

Si $f\varepsilon T_\Omega$, avec (1) $f(t)\sim\sum_{-\infty}^{\infty}a_k e^{\lambda_k t}$ ($\lambda_k\varepsilon\Omega$), est nulle sur un intervalle de longueur $d$ et si

$$\lim_{R\to\infty}[M_\Omega(R)-(d/\pi)R-\log R/q]=-\infty$$

($p^{-1}+q^{-1}=1$), alors $f(t)\equiv 0$ (presque partout). Supposons maintenant que tous les $a_k$ soient réels. En écrivant les $\lambda_k$, situés dans $(-t, t)$, dans l'ordre croissant, écrivons dans le même ordre les coefficients $a_{-r}, \cdots, a_0, \cdots, a_s$. Désignons par $n_0(t)$ le nombre de changements de signe de ces coefficients, et formons la fonction correspondante $N_0(R)$. Si $f\varepsilon T_r$ avec (1), si $f(t)=0$ sur $(-d/2, d/2)$ et si

$$\liminf[N_0(R)-Rd/\pi+\log R/p]=-\infty,$$

alors $f(t)\equiv 0$ (presque partout). Un théorème permet de donner la forme des coefficients de la fonction lorsqu'on sait qu'elle s'annulle sur un intervalle assez long.

*S. Mandelbrojt* (Houston, Tex.).

**Pollaczek, F. Application d'opérateurs intégro-combinatoires dans la théorie des intégrales multiples de Dirichlet.** Ann. Inst. H. Poincaré **11**, 113–133 (1949).

The author has spent years on research (connected with automatic telephone exchanges) which requires the evaluation of complicated integrals in $n$ dimensions, where $n$ is large and the domain of integration is determined by combinatorial relations such as the condition that $\nu$ out of the $n$ variables satisfy certain inequalities. He uses Fourier integrals extensively, and he has developed a special symbolism of an operational calculus. The basic ideas are here explained in a systematic way without reference to applications or to previous work (which will now be more readily accessible). The symbolism and manipulations are too complicated to be explained in a review but the idea may be illustrated as follows.

Let $s(x)=1$ or 0 according as $x>0$ or $x<0$, and use the letter $s$ as an operator. If $x_1, \cdots, x_n$ are given numbers denote by $\min^{(m)} x_\nu$ the $m$th among them when ordered according to increasing magnitude. Since $s$ is expressible by a Dirichlet integral, $s\min^{(m)} x_\nu$ is given by an $n$-tuple Fourier integral. This is represented as a sum of certain products of elementary integral operators

$$C=(2\pi i)^{-1}\int_{-i\omega+\epsilon_\nu}^{i\omega+\epsilon_\nu}\cdots z_\nu^{-1}dz_\nu,$$

so that the final expression takes the form

$$s\min^{(m+1)} x_\nu=S_n^{(m)}\exp\sum_1^n x_\nu z_\nu.$$

The operators $S_n^{(m)}$ (and other more complicated ones) thus stand for a combinatorial relation ("at most $m$ among the $x_\nu$ are negative") and their use appears to simplify matters considerably. [For applications to integral equations cf. the companion paper, same vol., 135–173 (1949); these Rev. 11, 672.]

*W. Feller* (Princeton, N. J.).

**Fränz, Kurt. Relations between signal and spectrum.** Revista Unión Mat. Argentina **14**, 140–155 (1950). (Spanish. German summary)

The author's results are as follows. Let

$$f(x)=\int_{-\tau/2}^{\tau/2}u(t)e^{-ixt}dt,$$

where $u(t)$ is real and piecewise continuous; let

$$E=\int_{-\tau/2}^{\tau/2}u^2dt=(2\pi)^{-1}\int_{-\infty}^{\infty}|f(x)|^2dx;$$

$$E(\Omega)=(2\pi)^{-1}\int_{-\Omega}^{\Omega}|f(x)|^2dx.$$

(1) For $-\infty<x<\infty$,

$$|f(x)|^2\leqq\tau E/\{1+[1-x^{-1}\tau^{-1}|\sin xt|]^2\}.$$

(2) If $u(t)\geqq 0$, then $|f(x)|\geqq f(0)\cos\frac{1}{2}x\tau$ for $|x|<\pi/\tau$.
(3) If $u(t)\geqq 0$ and $u'(t)\operatorname{sgn} t\geqq 0$, then

$$|f(x)|\geqq 2f(0)x^{-1}\tau^{-1}\sin\tfrac{1}{2}xt$$

for $|x|\leqq\pi/\tau$. (4) The maximum of $E(\Omega)$ for given $\tau$ and $E$ is attained for a solution of

$$u(t)=\pi^{-1}\lambda\int_{-\tau/2}^{\tau/2}(t-s)^{-1}u(s)\sin\Omega(t-s)ds,$$

and is the reciprocal of the first characteristic value of this integral equation; some numerical results are given. (5) If $T(s)=\int_0^\infty u'(t)e^{-st}dt$ then, if $T$ is small enough at $\infty$, we have $2\pi|u'(t)|\leqq\int_{-\infty}^{\infty}|T(iy)|dy$. (6) If $u'(t)$ is nonnegative and small enough at $\infty$, then $|T^{(n)}(s)|$ and $|D^{2n}|T|^2|$, $D=d/ds$, attain their maxima at $s=0$. The author expresses his results in the language of communication engineering; thus (1)–(4) show that for a signal (of simple form) of length $\tau$ the spectrum, in $|x|<\pi/\tau$, depends mainly on $\tau$ and little on the form of the signal.

*R. P. Boas, Jr.*

**Mandelbrojt, S., et Agmon, S. Une généralisation du théorème tauberien de Wiener.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 167–176 (1950).

Suppose that $\varphi_0(x)$ is bounded and measurable, $K_0(x)$ and $K(x)$ belong to $L(-\infty, \infty)$ and that

$$\int_{-\infty}^{\infty}\phi_0(x)K_0(y-x)dx=0,\quad \phi(x)=\int_{-\infty}^{\infty}\phi_0(t)K(x-t)dt.$$

Then it is shown that the function $F(z)$ defined by $F(z)=\int_{-\infty}^{0}\phi(t)e^{-izt}dt$ for $I(z)>0$ is regular in the whole complex plane except for a perfect subset $\Sigma$ of the intersection $I(K_0, K)$ of the frontier sets of the sets $\Omega(K_0)$, $\Omega(K)$ of real zeros of the Fourier transforms of $K_0$, $K$, respectively. Moreover, if $\Omega(K)\supset\Omega(K_0)$ and $I(K_0, K)$ is countable, then for any $\epsilon>0$ it is possible to define an integer $N$ and real $a_\nu$, $\xi_\nu$ so that

$$\int|K(x)-\sum_1^N a_\nu K_0(x-\xi_\nu)|dx<\epsilon,$$

a result which includes Wiener's general Tauberian theorem. The countability condition on $I(K_0, K)$ may be omitted if certain extra conditions are imposed on $K(x)$ and $K_0(x)$ or, in particular, if $xK(x)$ belongs to $L$.

*H. R. Pitt.*

**Heins, Albert E. A note on a singular integral equation.** Proc. Cambridge Philos. Soc. **46**, 268–271 (1950).

By means of the complex Fourier transform technique of Paley and Wiener, an explicit solution of the Wiener-Hopf

type equation

$$\phi(\xi)=\frac{\lambda}{2}\int_0^\infty \frac{\phi(\eta)d\eta}{\cosh\frac{1}{2}(\xi-\eta)} \qquad (\xi>0)$$

is found in the form $\phi(\xi)=Be^{\xi}P_{\beta-1}(e^{\xi})$, where $B$ is an arbitrary constant, $\lambda\pi=\cos\pi\beta$ and $P_{\beta-1}(e^{\xi})$ is the Legendre function of the first kind. *H. R. Pitt* (Belfast).

**Varma, R. S. An inversion formula for the generalised Laplace transform.** Proc. Edinburgh Math. Soc. (2) **8**, 126–127 (1949).

The author derives a complex inversion formula for the integral transform

$$\phi_m^k(s)=\int_0^\infty (2st)^{-1}W_{k,m}(2st)f(t)dt,$$

where $W_{k,m}$ is Whittaker's function.

*I. I. Hirschman, Jr.* (St. Louis, Mo.).

**Goodspeed, F. M. The relation between functions satisfying a certain integral equation and general Watson transforms.** Canadian J. Math. **2**, 223–237 (1950).

Let $F(x)\varepsilon L_2(0,\infty)$ satisfy the equation

$$\int_0^\infty F(x)F(ux)dx=1/(u+1).$$

It is shown that the function $G(x)$ is effectively defined by the formula

$$G(x)=\lim_{v\to 0+}(2\pi)^{-1}\int_0^x [F(u+iv)+F(u-iv)]dv$$

and that for $A(x)\varepsilon L_2(0,\infty)$ the relations

$$(d/dy)\int_0^\infty A(x)G(xy)x^{-1}dx=B(y),$$

$$(d/dx)\int_0^\infty B(y)G(xy)y^{-1}dy=A(x)$$

are consequences of one another. These formulas were stated by Ramanujan in a slightly different form. As originally given they were not always correct.

*I. I. Hirschman, Jr.* (St. Louis, Mo.).

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### *Polynomials, Polynomial Approximations*

**Meĭman, N. N. Some problems on the distribution of the zeros of polynomials.** Uspehi Matem. Nauk (N.S.) **4**, no. 6(34), 154–188 (1949). (Russian)

This is a treatment of distribution problems concerning the zeros of polynomials, with special reference to the Hurwitz problem of determining when all the zeros lie in a half-plane, the desideratum being that the solution to each problem be effected by an algorithm, that is, by a finite number of rational operations on the coefficients of the polynomial. In the sense that much of the work goes over old ground the approach is largely expository, although some new features are present (particularly in § 10, to be referred to later).

Sections 1–3 deal with miscellaneous results, not new and not germane to the above-mentioned program, so they may be passed over. The real purpose of the article begins with § 4, where (generalized) Sturm sequences are taken up. The ordered set of real functions (4.1) $V_0(x),\cdots,V_m(x)$ is a Sturm sequence if (i) adjacent functions have no common real zero, and (ii) at each real zero of $V_j(x)$ $(j\neq 0,m)$ the neighboring functions $V_{j-1}(x)$, $V_{j+1}(x)$ are of opposite sign. A real zero of $V_0(x)$ is of first (second) category if $V_0(x)V_1(x)$ changes through this zero from minus to plus (from plus to minus); and for a zero of $V_m(x)$, category is defined in the opposite way. Let $W(x)$ be the number of sign changes in sequence (4.1). For all $a, b$ that are not zeros of either $V_0$ or $V_m$, $W(a)-W(b)=(\alpha_1-\alpha_2)-(\beta_1-\beta_2)$, where $\alpha_i, \beta_i$ $(i=1,2)$ are the number of zeros (not counting multiplicity) of $i$th category of $V_0$ and $V_m$ on the interval $(a, b)$.

In terms of these categories of zeros a solution is given in § 5 of the Hurwitz problem, using for simplicity the upper and lower half-planes of the complex plane. [The final result is essentially that of theorem (37, 1), page 129, in M. Marden, The Geometry of the Zeros of a Polynomial in a Complex Variable, Amer. Math. Soc., New York, 1949 [these Rev. 11, 101]; chapters IX and X contain some material identical with, or closely related to, that found in the present work.] Sections 6, 7 deal with the Sturm sequence generated by two real polynomials $g(z)$, $h(z)$, without common real roots, by means of the Euclidean algorithm for H.C.F.: $V_0=g$, $V_1=h$; and $V_2,\cdots,V_m$ (=constant) determined by $V_0=V_1Q_1-V_2,\cdots,V_{m-2}=V_{m-1}Q_{m-1}-V_m$, $V_{m-1}=V_mQ_m$. Let $\delta_j=\pm 1$ $(j=1,\cdots,m)$ according as the leading coefficient of $Q_j$ is positive or negative, and let degree $Q_j=r_j$. Then $\alpha_1-\alpha_2=\sum_j'\delta_j$, summed over all $j$ for which $r_j$ is odd. Here $\alpha_j$ $(j=1,2)$ is the number of zeros of $g(z)$ of $j$th category relative to the above Sturm sequence. An algorithm attributed to Yu. I. Neĭmark [Doklady Akad. Nauk SSSR (N.S.) **58**, 357–360 (1947); these Rev. **9**, 348] simplifies the determination of the quantities $\delta_j, r_j$; and this leads also to a solution of the Hurwitz problem, as well as to an expression for the number of real zeros of a polynomial.

Section 8 deals with a nonalgorithmic treatment of the Hurwitz problem [for this see Marden, op. cit., chapter IX]; and § 9 relates information concerning zeros of the polynomial $F(z)=g(z)+ih(z)$ (assumed to lie in the upper half plane) to the zeros (all real) of $g(z)$ and $h(z)$. The problem of the zeros inside a circle can be reduced to the Hurwitz problem of zeros in a half-plane, but can also be treated independently by means of an algorithm, as was considered by Schur and by Cohn [this is discussed in Marden, op. cit., chapter X]. The present work contains in § 10 a different algorithm for the same purpose. Let $f(z)=a_0z^n+\cdots+a_n$, $f^*(z)=\bar{a}_nz^n+\cdots+\bar{a}_0$, and set $w(z)=f(z)/f^*(z)$. Choose $z_0$ in $|z|<1$, with $|w(z_0)|\neq 1$, and set

$$\zeta(z)=\{w(z_0)-w\}\{1-\overline{w(z_0)}w\}^{-1},$$
$$w_1(z)=\{1-\bar{z}_0z\}\{z_0-z\}^{-1}\zeta(z);$$

take $z_1$ in $|z|<1$, with $|w_1(z_1)|\neq 1$ and let

$$\zeta_1(z)=\{w_1(z_1)-w_1\}\{1-\overline{w_1(z_1)}w_1\}^{-1},$$
$$w_2(z)=\{1-\bar{z}_1z\}\{z_1-z\}^{-1}\zeta_1(z);$$

and so on, obtaining a sequence (a) $w_0=w, w_1,\cdots,w_r$ ($w_r=$ constant, $|w_r|=1$). If $r=n$, then the number of zeros of $f(z)$ of magnitude less than unity is given by $i_0+(i_2-i_1)+(i_4-i_3)+\cdots+(i_{k-1}-i_{k-2})$ for $k$ odd and by $n-[(i_1-i_0)+(i_3-i_2)+\cdots+(i_{k-1}-i_{k-2})]$ for $k$ even. Here $i_0,\cdots,i_{k-1}$ are those indices $i$ in sequence (a) for which $|w_i(z_i)|>1$. *I. M. Sheffer* (State College, Pa.).

**Throumoulopoulos, Laz. E. On the modulus of the roots of polynomials.** Bull. Soc. Math. Grèce **24**, 68–73 (1949). (Greek. French summary)

Let $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$, $r_1,\cdots,r_n$ its roots, and $r=\max\{|r_1|,\cdots,|r_n|\}$. For every positive integer $k$, let

$p_k(z)=z^n+a_{1k}z^{n-1}+\cdots+a_{nk}$ be the polynomial with roots $r_1^k, \cdots, r_n^k$ and define $R_k=\sum_{i=1}^n|a_{ik}|^{1/i}$. Then it is shown that (a) $\lim_{k\to\infty}[R_k]^{1/k}=r$ and (b) for every pair of integers $l, m, r\leqq[R_{lm}]^{1/(lm)}<r\sqrt{2}$. These results follow from Walsh's theorem ($r^k\leqq R_k$) and the expression of each $a_{ik}$ in terms of the roots, giving $R_k<r^k[\sum_{i=1}^n\binom{n}{i}^{1/i}]$. The paper also contains a generalization of Walsh's theorem as follows: if $f(z,u)=u^n+a_1(z)u^{n-1}+\cdots+a_n(z)=0$ defines an algebraic function $\varphi(z)$ with $n$ branches, $m(r)$ is the maximum of all branches on $|z|=r$, and $M_i(r)=\max_{|z|=r}|a_i(z)|$, then $m(r)\leqq\sum_{i=1}^n[M_i(r)]^{1/i}$. *J. Dugundji.*

**Gatteschi, Luigi. Approssimazione asintotica degli zeri dei polinomi ultrasferici.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 399–411 (1949).

In a recent paper [Boll. Un. Mat. Ital. (3) **4**, 240–250 (1949); these Rev. **11**, 432] the author studied the zeros of Legendre polynomials. The present paper deals with the corresponding, much more complicated, question for the ultraspherical polynomials $P_n^{(\lambda)}(\cos\theta)$, $0<\lambda<1$. Using the asymptotic representation of $P_n^{(\lambda)}$ due to Stieltjes (with three terms) he proves that for the $r$th zero $\theta_r$ of this function the following formula holds:

$$\theta_r=\alpha_r+\frac{\lambda(1-\lambda)(n+\lambda+3)}{2(n+\lambda)(n+\lambda+1)(n+\lambda+2)}\cot\alpha_r+\rho(n,\lambda),$$

where $\alpha_r=\frac{1}{2}(2r+\lambda-1)\pi/(n+\lambda)$,

$$|\rho(n,\lambda)|<(23.7+8.7C)/16(n+\lambda)^4.$$

Here $C=\cot^2\frac{1}{2}\pi\lambda$ for $0<\lambda\leqq\frac{1}{2}$, $C=\cot\frac{1}{2}\pi\lambda$ for $\frac{1}{2}\leqq\lambda<1$; moreover $n>N(\lambda)$ where $N$ can be obtained in explicit terms. Applying the general formula to Legendre polynomials, for the fifth zero $\xi$ of $P_{16}(x)$ the inequalities $0.61784<\xi<0.61791$ follow. The true value of $\xi$ is 0.61787. *G. Szegö* (Stanford University, Calif.).

**Szász, Otto. Inequalities concerning ultraspherical polynomials and Bessel functions.** Proc. Amer. Math. Soc. **1**, 256–267 (1950).

The purpose of this paper is to refine certain results of a note of the reviewer [Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429] extending an inequality of Turán to ultraspherical polynomials $P_n^{(\lambda)}(x)$. Writing

$$F_n(x)=P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1),\ (n+\lambda)(n+2\lambda-1)F_n^2(x)$$
$$-(n+\lambda-1)(n+2\lambda)F_{n-1}(x)F_{n+1}(x)=D_n^{(\lambda)}(x),$$

he proves that

$$\frac{\lambda(\lambda+1)\Gamma(n+2\lambda)}{(n+\lambda-1)(n+\lambda)\Gamma(2\lambda+1)\Gamma(n)}D_n^{(\lambda)}(x)$$

is monotonically decreasing as $n\to\infty$; $0<\lambda<1$, $-1\leqq x\leqq 1$. The limit $\chi(x)$ is a nonnegative function assuming its minimum at $x=1$. This results in certain inequalities for $F_n^2-F_{n-1}F_{n+1}$. Similar inequalities are obtained for Hermite polynomials and for Bessel functions. *G. Szegö.*

**Rymarenko, B. A. On polynomials which are monotonic on the whole real axis.** Doklady Akad. Nauk SSSR (N.S.) **71**, 1029–1032 (1950). (Russian)

The author states without proof the solutions of the problems of finding the smallest variation on $[-1, 1]$, and the minimizing polynomial, for polynomials of odd degree $n$ which are monotone increasing on the whole real axis and for which either (1) the coefficients of $x^n$, $x^{n-1}$, $x^{n-2}$ are given; (2) the coefficient of $x^n$ and the derivative at a prescribed point are given; or (3) the first two derivatives are given at 1. The solutions are too complicated to reproduce here. *R. P. Boas, Jr.* (Evanston, Ill.).

**Zuhovickiĭ, S. I., and Kreĭn, M. G. Remark on a possible generalization of the theorems of A. Haar and A. N. Kolmogorov.** Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 217–229 (1950). (Russian)

Let $Q$ be a compact point set, and $F_1(q), F_2(q), \cdots, F_n(q)$ be real-valued continuous functions on $Q$. For every function $\Phi(q)$ on $Q$ there is a "polynomial" $\sum\alpha_jF_j(q)$ of best approximation. A. Haar showed that this polynomial is unique for every continuous $\Phi(q)$ if and only if the system $\{F_j(q)\}$ is a Chebyshev system, i.e., if $\sum\gamma_jF_j(q)$ can vanish at $n-1$ points at most, provided $\sum|\gamma_j|\neq 0$ [Math. Ann. **78**, 294–311 (1917)]. This result was extended by Kolmogoroff [Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 216–221 (1948); these Rev. **10**, 35] to the case of approximation of complex-valued functions by complex-valued polynomials. In the present note the authors generalize the result to vector functions and to hypercomplex functions. *A. Zygmund.*

**Merli, Luigi. Sulla rappresentazione delle funzioni continue con una classe di polinomi interpolanti del tipo di Lagrange.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 212–216 (1950).

Soit $x_k^{(n)}=\cos(2k-1)\pi/2n$, $1\leqq k\leqq n$, $n=1, 2, \cdots$, et $l_k^{(n)}(x)$ le polynome de degré $(n-1)$ qui prend la valeur 1 pour $x_k^{(n)}$ et s'annule pour $x_j^{(n)}$ $(j\neq k)$; alors, si $f(n)$ est continue pour $-1\leqq x\leqq 1$, $\sum_1^n f(x_{2k-1}^{(2n)})\{l_{2k-1}^{(2n)}(n)+l_{2k}^{(2n)}(n)\}$ tend uniformément vers $f(n)$ lorsque $n$ augmente indéfiniment. *J. Favard* (Paris).

**Sansone, Giovanni. Su una disuguaglianza relativa ai polinomi di Legendre.** Boll. Un. Mat. Ital. (3) **4**, 339–341 (1949).

Referring to a previous note [same vol., 221–223 (1949); these Rev. **11**, 432] the author is interested in the behavior of $\Delta_n(x)=[P_n(x)]^2-P_{n-1}(x)P_{n+1}(x)$ for $x>1$. This function is negative and decreasing to $-\infty$ as $x\to\infty$. Also $P_{n+1}(x)/P_n(x)<x+(x^2-1)^{\frac{1}{2}}$ and for the ratio of these quantities an asymptotic expression is given as $n\to\infty$. The constants involved in this expression are specified. *G. Szegö* (Stanford University, Calif.).

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## *Special Functions*

**Horton, C. W. On the extension of some Lommel integrals to Struve functions with an application to acoustic radiation.** J. Math. Physics **29**, 31–37 (1950).

Investigation of the integrals $C_n(z)=\int_0^z z^nJ_n(z)dz$ and $D_n(z)=\int_0^z z^nH_n(z)dz$ where $J_n$ denotes the Bessel function of the first kind and $H_n$ Struve's function. Recurrence relations, (inhomogeneous) differential equations, and reduction formulae are obtained. The author also gives reduction formulae for some related integrals for which the integrand contains either Lommel's function $S_{m,n}$, or else a product of two Bessel functions, two Struve functions, or else a product of a Bessel function with a Struve function. The application of the results to a problem in acoustic radiation is briefly indicated. *A. Erdélyi* (Pasadena, Calif.).

Le cercle polaire d'1 tore est le cercle conjugué commun à tous les cercles méridiens, et non le lieu du centre des sphères inscrites (comme le rapporteur l'a écrit par inadvertence.)



**Luke, Yudell L. Some notes on integrals involving Bessel functions.** J. Math. Physics **29**, 27–30 (1950).

Integrals of the form $G(m, n, x, y) = \int_a^x e^{kx} u^m Z_n(uy)du$ in which $Z_n$ denotes any of the Bessel functions or modified Bessel functions occur in many problems of applied mathematics. The author derives a reduction formula for such integrals, gives an expansion in powers of $1-y^2$, and gives an explicit evaluation for $G(n, n, x, 1)$ together with a few particular cases (with $a=0$) of this last result.

*A. Erdélyi* (Pasadena, Calif.).

**Sips, Robert. Convergence des séries représentant les fonctions de Mathieu et les fonctions d'onde sphéroïdales.** Bull. Soc. Roy. Sci. Liège **18**, 498–515 (1949).

Verf. geht von der Fourier-Entwicklung für die Mathieuschen Funktionen aus und stellt für die höheren Glieder dieser Entwicklungen die Differentialgleichungen auf, welche er sukzessive löst. Er untersucht sodann die Konvergenzbedingungen der erhaltenen Reihen. *M. J. O. Strutt.*

**Buchholz, H. Bemerkung zur Fourierschen Reihe für die dreidimensionale Greensche Funktion der Wellengleichung des unbegrenzten Raums in den Koordinaten des Drehparabols.** Z. Angew. Math. Mech. **30**, 125–127 (1950).

In an earlier paper [Z. Physik **124**, 196–218 (1948); these Rev. **10**, 453] the author has discussed the representation in coordinates of the paraboloid of revolution of a spherically symmetric wave function whose centre does not coincide with the focus of the confocal system of paraboloids. The representation was of the form of a Fourier cosine series with coefficients which are infinite integrals involving products of confluent hypergeometric functions. In the present paper this expression is transformed and now appears as a complex Fourier series (of exponential functions). The coefficients are again infinite integrals involving products of confluent hypergeometric functions, but the products involved in the new representation are different from the old ones. He also derives a few consequences of his new representation which establish certain connections between cylindrical and paraboloidal wave functions.

*A. Erdélyi* (Pasadena, Calif.).

**Lorenz, Paul. Eine Herleitung der Stirlingschen Formel**

$$n! = n^n\sqrt{\pi 2n}e^{-n+\frac{n^{-1}}{12}-\frac{n^{-3}}{360}+\frac{n^{-5}}{1260}-\vartheta_5(n)\frac{n^{-7}}{1680}}$$

**mit kleinstem Aufwand.** Arch. Math. **2**, 222–226 (1950).

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## *Harmonic Functions, Potential Theory*

**Kodama, Sikazô. L'inégalité nécessaire à la théorie de la classe quasi-analytique de fonctions de deux variables.** Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **24**, 83–88 (1944).

If $g(u, v)$ is holomorphic in $|u|\leqq 1$, $|v|\leqq 1$, $g(0, 0)\neq 0$, then

$$\log|g(0,0)| < \tfrac{1}{4}\pi^{-2}\int_0^{2\pi}\int_0^{2\pi}\log|g(e^{i\alpha}, e^{i\beta})|\,d\alpha d\beta.$$

Imitation of the classical proof for one variable.

*S. Mandelbrojt* (Houston, Tex.).

**Cotton, Émile. Sur la représentation asymptotique du potentiel newtonien.** Ann. Inst. Fourier Grenoble **1** (1949), 13–25 (1950).

The author studies first the Newtonian potential due to a particle, obtaining an asymptotic expansion by a method depending on the use of symbolic forms; this is then generalized for a simple distribution. A parallel theory is developed for a double distribution on a surface, involving the solid angle subtended by the boundary. Finally, the author correlates some of this material with classical results.

*F. W. Perkins* (Hanover, N. H.).

**Pogorzelski, Witold. Sur les propriétés du potentiel retardé.** Prace Mat.-Fiz. **47**, 61–66 (1949).

Enoncé et démonstration de propriétés classiques des potentiels retardés (potentiels de volume ou de double couche superficielle). *L. Schwartz* (Nancy).

**Fichera, Gaetano. Proprietà di media toroidali delle funzioni armoniche.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 431–435 (1949).

Soit $D$ le domaine borné de $R^3$ limité par un tore $T$ d'axe $Oz$, de cercle polaire $C$ (lieu des centres des sphères inscrites); il existe une fonction $\mu(P)$ définie sur $T$ telle que, pour toute fonction $u$ continue sur $D\cup T$ et harmonique dans $D$ on ait:

$$\int_C u(Q)ds_Q = \int\int_T \mu(P)u(P)d\sigma_P$$

où $ds$ est l'élément d'arc de $C$, $d\sigma$ l'élément d'aire de $T$ (l'existence et l'unicité de $\mu$ résultent de théorèmes généraux; la mesure $\mu d\sigma$ n'est autre que la "balayée" sur $T$ de la mesure $ds$, portée par $C$). L'auteur donne une expression simple de $\mu$ à l'aide de la fonction de Legendre $Q_{-\frac{1}{2}}$; sa méthode utilise l'expression du Laplacien en coordonnées "toroïdales." Il montre également pour les fonctions $u$ harmoniques dans l'extérieur de $T$, régulières à l'$\infty$, la propriété de moyenne:

$$\int_{-\infty}^{\infty} U(z)(1+z^2)^{-\frac{1}{2}}dz = \int\int_T \nu(P)u(P)d\sigma_P,$$

$U(z)$ étant la valeur prise par $u$ en un point de l'axe $Oz$, et $\nu$ s'exprimant à l'aide de la fonction de Legendre $P_{-\frac{1}{2}}$.

*J. Deny* (Strasbourg).

**Rudin, Walter. Integral representation of continuous functions.** Trans. Amer. Math. Soc. **68**, 278–286 (1950).

L'auteur considère dans le plan (avec extension immédiate à l'espace à $n\geqq 2$ dimensions) et dans l'ordre d'idée des laplaciens généralisés, la limite supérieure $\Lambda^*F$ et la limite inférieure $\Lambda_*F$ de $4r^{-2}[M_{F^r}(T)-F(T)]$ où $M_{F^r}$ est la moyenne de $F$ sur la circonférence de centre $T$ et rayon $r$. Approfondissant la représentation intégrale de Riesz des fonctions sousharmoniques $u$, il cherche quand la distribution de masses associée est définie par une simple densité et montre qu'il en est ainsi lorsque $u$ est finie continue et que $\Lambda_* u < +\infty$ hors d'un ensemble fermé de capacité nulle; la densité est alors le laplacien généralisé (valeur commune presque partout de $\Lambda^*$ et $\Lambda_*$). En fait l'auteur étudie plus généralement la représentation analogue de $u$ continue non nécessairement sousharmonique en se basant sur un examen du laplacien généralisé du potentiel d'une couche de densité $f$.

*M. Brelot* (Grenoble).

**Aĭzenštat, N. D. On a class of linear operators.** Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 23–36 (1948). (Russian)

The article deals with the following generalization of the Dirichlet problem. Every interior point $P$ of a bounded, connected, $n$-dimensional region $G$ is the center of a sphere $\omega(P)$, whose radius $h(P)$ is continuous in $P$ and tends to zero for points near the boundary $\Gamma$. A suitably normalized function $A(P)$, regular and strictly positive in $G+\Gamma$, defines the linear operator $L_A^n f = \int_\omega A(M) f(M) d\omega_M$. For a function $F$, continuous on the boundary $\Gamma$, a function $u_F(P)$ is sought such that (i) in $G$, $u_F(P) = L_A^n u_F(P)$,

$$\text{(ii)} \qquad \lim_{P \to Q} u_F(P) = F(Q) \qquad (P \in G,\ Q \in \Gamma).$$

The method used by the author is a generalization of the method of Perron [Math. Z. **18**, 42–54 (1923)]. A function $v$ is $A$-superharmonic if, for a given system of spheres, $v(P) \geqq L_A^n v(P)$; an upper function $v_F$ for a given boundary problem is an $A$-superharmonic function for which $\liminf_{P \to Q} v_F(P) \geqq F(Q)$. Lower functions are analogously defined in terms of $A$-subharmonic functions. The author shows that the infimum $u_F$ of the set of upper functions is $A$-harmonic (that is, satisfies (i) above) and is equal to the supremum of the lower functions. He gives sufficient conditions for the regularity of a boundary point (a boundary point $Q$ is regular if condition (ii) above is satisfied there). He shows that a point which is regular for Wiener's construction [J. Math. Physics **3**, 24–51 (1924)] (as applied to the problem above) fulfills the sufficient conditions given here. *M. Gottlieb* (Chicago, Ill.).

***Künzi, Hans Paul. Der Fatou'sche Satz für harmonische und subharmonische Funktionen in $n$-dimensionalen Kugeln.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1949. 27 pp.

Démonstration du théorème suivant. Soit $V$ une fonction sousharmonique dans le domaine constitué par la boule unité de l'espace $R^p$ des $p$ variables réelles, avec l'hypothèse que l'integrale $\int V(P) d\sigma$, prise sur la sphère concentrique de rayon $r$, demeure bornée quand $r \to 1$. Alors $V$ a une limite radiale finie sur presque tout rayon de la sphère unité quand on tend vers celle-ci. L'auteur manifestement n'avait pas connaissance des deux memoires de Privaloff [Rec. Math. [Mat. Sbornik] N.S. **3(45)**, 3–25 (1938); Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **1938**, 191–220]; le premier d'entre eux contient les résultats de l'auteur avec la même méthode de démonstration.

*P. Lelong* (Lille).

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## *Differential Equations*

**Germay, R. H. J. Sur l'intégrale non singulière de l'équation différentielle du premier ordre de forme non résolue, tangente à l'intégrale singulière de première espèce de cette équation.** Bull. Soc. Roy. Sci. Liège **18**, 334–342 (1949).

Let the function $F(x, y, z)$ be analytic at $(x_0, y_0, z_0)$, and such that $F_z(x_0, y_0, z_0) \neq 0$. Suppose the differential equation (*) $F(x, y, y') = 0$ admits a singular integral $y = \phi(x)$ such that $y_0 = \phi(x_0)$, $\phi'(x_0) = z_0$. The paper presents a method of successive approximations for finding a nonsingular integral $y = \psi(x)$ of (*) such that $\psi(x_0) = \phi(x_0)$, $\psi'(x_0) = \phi'(x_0)$.

*F. G. Dressel* (Durham, N. C.).

**Horošilov, V. V. On the solutions of systems of differential equations with an irregular singular point.** Doklady Akad. Nauk SSSR (N.S.) **72**, 241–242 (1950). (Russian)

A special system of ordinary differential equations of the form $dY/dt = Y[P_0 + P_1 t^{-1} + P_2 t^{-2} + \cdots]$ is considered. Here $Y$ and $P_k$ are two-by-two matrices; the $P_k$ are constant. The problem is divided into three cases depending upon the form of $P_0$. For $P_0$ a diagonal matrix with nonzero elements $a_1$, $a_2$, case I is $\Re(a_1) \neq \Re(a_2)$ and case II is $\Re(a_1) = \Re(a_2)$. In case III,

$$P_0 = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}.$$

The general form of the fundamental system of solutions is exhibited in each case. It is pointed out that this process, due to Erougin [Trav. Inst. Math. Stekloff **13** (1946); these Rev. **9**, 509] is also applicable to second-order equations. The equations of Whittaker and Bessel are given as examples.

*C. G. Maple* (Washington, D. C.).

**Bruwier, L. Sur l'intégration des systèmes normaux d'équations différentielles par une méthode d'approximations successives.** Bull. Soc. Roy. Sci. Liège **18**, 347–356 (1949).

Let $i = 1, 2, \cdots, n$ and $m = 0, 1, \cdots$. Consider the successive approximations for the system $y_i' = f_i(t, y_1, \cdots, y_n)$ defined by the quadratures or integral equations

$$y_i^{m+1}(t) = y_i^0 + \int_0^t f(x, u_1^m(s), \cdots, u_n^m(s))\,ds,$$

where $u_i^m(s)$ is either $y_i^m(s)$ or $y_i^{m+1}(s)$ (and the choice is an arbitrary function of $i$ and $m$). If the functions $f_i$ are continuous and satisfy a Lipschitz condition with respect to $y_1, \cdots, y_n$, then the successive approximations converge to a solution on the standard $t$-range. The proof depends on an estimate for a function $u(t) \geqq 0$ satisfying $u'(t) \leqq au(t) + bv(t)$, where $a$, $b$ are nonnegative constants and $v(t) \geqq 0$ is a given function; the desired estimate follows at once from standard procedures [cf. Kamke, Differentialgleichungen reeler Funktionen, Akademische Verlagsgesellschaft, Leipzig, 1930, p. 93].

*P. Hartman* (Baltimore, Md.).

**Szarski, Jacek. On an oscillatory property of successive approximations.** Ann. Soc. Polon. Math. **22** (1949), 201–206 (1950).

Let $f^i(t, y^1, \cdots, y^n)$ be continuous functions for $a \leqq t \leqq b$ and arbitrary $(y^1, \cdots, y^n)$ with the property that for some $k$, where $1 \leqq k \leqq n$, the functions $f^i$ and $-f^j$ for $i = 1, \cdots, k$ and $j = k+1, \cdots, n$ are nonincreasing functions of each of the variables $y^1, \cdots, y^k$ and $-y^{k+1}, \cdots, -y^n$. Let $(y^1(t), \cdots, y^n(t))$ be a solution of the system $y^{i\prime} = f^i$ for $i = 1, \cdots, n$ and $a \leqq t \leqq b$. Finally, let

$$y^{h,\nu}(t) = y^h(a) + \int_a^t f(u, y^{1,\nu-1}(k), \cdots, y^{n,\nu-1}(u))\,du,$$

where $h = 1, \cdots, n$; $\nu = 1, 2, \cdots$; and $y^1{}_0(t), \cdots, y^n{}_0(t)$ are continuous functions for $a \leqq t \leqq b$ with the properties that the differences $y^i(t) - y^i{}_0(t)$ and $y^i{}_0(t) - y^j(t)$ are of the same constant sign, say $\epsilon$ $(= +1$ or $-1)$, for $i = 1, \cdots, k$ and $j = k+1, \cdots, n$. Then for $\nu = 0, 1, \cdots$, the differences $y^i(t) - y^i{}_\nu(t)$ and $y^j{}_\nu(t) - y^j(t)$ are of the same constant sign $(-1)^\nu \epsilon$ for $i = 1, \cdots, k$ and $j = k+1, \cdots, n$. This result generalizes one of Wintner's [Mat. Tidsskr. A. **1948**, 22–24; these Rev. **10**, 455] dealing with the equation $y'' + f(t)y = 0$.

*P. Hartman* (Baltimore, Md.).

**Mitrinovitch, Dragoslav. À propos d'une note de M. D. Pompeiu relative à l'équation de Riccati.** Acad. Roum. Bull. Sect. Sci. 30, 255–263 (1947).

Some conditions sufficient for the integrability of Riccati's differential equation are derived from the invariance of the equation under a transformation which is linear fractional in the dependent variable. *P. Franklin.*

**Chiellini, Armando. Sui sistemi di Riccati.** Rend. Sem. Fac. Sci. Univ. Cagliari 18 (1948), 44–58 (1949).

If $n+1$ functions $\xi_1, \cdots, \xi_{n+1}$ satisfy a system of $n+1$ ordinary linear homogeneous differential equations of the first order, their $n$ ratios $y_1, \cdots, y_n$ satisfy a nonlinear system of the first degree in the derivatives and quadratic in the unknowns. For $n=1$ such a system becomes a Riccati equation and for general $n$ Schlesinger has called such a system a Riccati system. The present paper gives formal properties of a Riccati system: (i) When the system is solved for the derivatives $y_i'$, every quadratic term $y_jy_k$ for which $i \neq j$, $i \neq k$ has zero coefficient in the equation for $y_i'$; (ii) linear transformation of the $y$'s gives a normal form in which the coefficients of all the other quadratic terms are different from zero and a reduced form in which they are unity; (iii) integration of the general linear homogeneous equation of the $(n+1)$th order is equivalent to the integration of a reduced Riccati system of $n$ equations; (iv) the knowledge of $n$ or $n-1$ independent solutions of a Riccati system of $n$ equations reduces its solution to quadratures or to the solution of a Riccati equation and quadratures, respectively. The Riccati systems of two equations reducible to linear systems with constant coefficients are specified.

*J. M. Thomas* (Durham, N. C.).

**Gosse, René. Sur une équation de Langmuir généralisée.** Ann. Inst. Fourier Grenoble 1 (1949), 5–11 (1950).

Making certain assumptions concerning the qualitative properties of the functions $p(x, y, y')$, $q(x)$, $a(y)$, and $f(y)$, this posthumous note studies some simple qualitative properties of the solutions of the differential equation

$$y''+y'p(x, y, y')+q(x)da(y)/dy=f(y).$$

*L. A. MacColl* (New York, N. Y.).

**Montaldo, Oscar. Esistenza di infiniti autovalori per un sistema differenziale lineare omogeneo a coefficienti costanti.** Rend. Sem. Fac. Sci. Univ. Cagliari 18 (1948), 59–71 (1949).

L'autore si occupa di sistemi del tipo

$$(p_i\lambda+1)y_i' = \sum_k^{1\cdots n} (q_{i,k}\lambda+q^*_{i,k})y_k \qquad (i=1, \cdots, n),$$

dove le $p_i$, $q_{i,k}$, $q^*_{i,k}$ son costanti e $\lambda$ è un parametro e studia l'esistenza di autovalori per il problema al contorno $y_i(b_i)=0$. Egli considera il caso che siano tutti nulli i numeri $p_1, \cdots, p_r$, $2 \leqq r < n$, e diversi da zero i determinanti delle due matrici $\|q_{i,k}\|$, $i, k=1, \cdots, n$, e $\|q_{j,h}\|$, $j, h=1, \cdots, r$, gli elementi principali di quest'ultima essendo inoltre o tutti nulli o tutti diversi da zero. *G. Scorza-Dragoni* (Padova).

**Wintner, Aurel. On free vibrations with amplitudinal limits.** Quart. Appl. Math. 8, 102–104 (1950).

The vector equation $dx/dt=A(t)x$ is considered where the matrix $A(t)$ is continuous for large $t$. Let $A'$ be the transpose of $A$ and let $\lambda(t)$ denote the least, and $\mu(t)$ the greatest, characteristic value of $\frac{1}{2}(A+A')$. If $\int^T\lambda(t)dt$ and $\int^T\mu(t)dt$ tend to finite limits as $T\to\infty$, then for every nontrivial solution $x(t)$, $|x(t)|$ tends to a finite nonvanishing limit as $t\to\infty$. Here $|x(t)|$ denotes the Euclidean length of the vector $x(t)$. *N. Levinson* (Cambridge, Mass.).

★**Rauch, Lawrence Lee. Oscillation of a third order nonlinear autonomous system. Contributions to the Theory of Nonlinear Oscillations,** pp. 39–88. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

The system considered is $\dot{x}=k_1[f(x)-(1+k_2)x-z]$, $\dot{y}=-k_3[f(x)-x-z]$, $\dot{z}=-k_4(y+z)$, where the $k_i$ are constants and $f(x)$ satisfies $|f(x)|<1$, $f(0)=0$, $xf(x)\geqq 0$. It is shown that under certain conditions on $f(x)$ a topological solid torus exists in $(x, y, z)$-space free of singular points and such that its boundary is cut from outside to inside by all characteristics which intersect it. A two-dimensional surface of section is determined in the torus and a periodic orbit established by the Brouwer fixed-point theorem. It is also shown that the torus contains all oscillatory solutions.

*N. Levinson* (Cambridge, Mass.).

**Malʹzel', A. D. On stability in the first approximation.** Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 171–182 (1950). (Russian)

Let $P(t)$ be an $n\times n$ matrix whose elements $p_{ik}(t)$ are continuous and bounded for $t\geqq t_0$; let $L(\mathbf{x}, t)$ be a vector function of the vector $\mathbf{x}=(x_1, x_2, \cdots, x_n)$, analytic in the $x_i$, vanishing together with its first partial derivatives at the origin for all $t\geqq t_0$. The system of differential equations (1) $d\mathbf{x}/dt=P(t)\mathbf{x}+L(\mathbf{x}, t)$ has $m$-fold stability at the origin if there is a manifold $M$ of dimension $\geqq m$ such that $\lim_{t\to\infty}\mathbf{x}(t)=0$ for all solutions $\mathbf{x}(t)$ having their initial values $\mathbf{x}(t_0)$ in $M$.

The author obtains a complicated but powerful sufficient condition for $m$-fold stability. Let the linear system $d\mathbf{x}/dt=P(t)\mathbf{x}$ possess $m$ independent solutions whose characteristic numbers $\lambda_1, \cdots, \lambda_m$ are positive. [Cf. A. Liapounoff, Problème Général de la Stabilité du Mouvement, Princeton University Press, Princeton, N. J., 1947, especially p. 224; these Rev. 9, 34.] Suppose that positive $k<\min(\lambda_1, \cdots, \lambda_m)$, and a constant $N$ exist such that for any continuous vector $\omega(t)$ whose components satisfy $|\omega_i(t)|<e^{-kt}$ the system $d\mathbf{y}/dt=(P(t)+kI)\mathbf{y}+\omega(t)$ has a solution $\mathbf{y}(t)$ whose components are bounded by $N$ for $t\geqq t_0$. Then (1) has $m$-fold stability at the origin. Following the formulation and proof of this theorem, it is shown that the author's stability criterion includes results of Liapounoff [op. cit., p. 254] and Malkin. *J. G. Wendel* (New Haven, Conn.).

**Letov, A. M. Strictly unstable regulating systems.** Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 183–192 (1950). (Russian)

The author considers the stability of the system

$$\dot{\eta}_k=\sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha+n_k\xi, \quad \dot{\xi}=f(\sigma), \qquad \sigma=\sum_{\alpha=1}^{n} p_\alpha\eta_\alpha-\xi$$

by the method of Lur'e [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 353–367 (1945); these Rev. 7, 300]. *J. G. Wendel* (New Haven, Conn.).

★**Diliberto, Stephen P. On systems of ordinary differential equations. Contributions to the Theory of Nonlinear Oscillations,** pp. 1–38. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

The reduction by means of linear transformations of linear systems of first order differential equations yields

such results as that there exists an orthogonal transformation such that if the coefficient matrix of the given system is bounded then so is that of the transformed triangular system. A nonsingular transformation also exists which transforms a given system to a diagonal one and takes a bounded coefficient matrix into a bounded one for the diagonal system. Other results are on geometric criteria for stability of closed orbits of nonlinear systems and on bounds on the number of periodic solutions of a system with polynomial right members. *N. Levinson* (Cambridge, Mass.).

***Stoker, J. J. Nonlinear Vibrations in Mechanical and Electrical Systems.** Interscience Publishers, Inc., New York, N. Y., 1950. xix+273 pp. $5.00.

This book is a well written exposition for physicists, engineers and applied mathematicians. The differential equations considered are either first order or second order. Chapter titles are: Linear vibrations; Free vibrations of undamped systems with nonlinear restoring forces; Free oscillations with damping and the geometry of integral curves; Forced oscillations of systems with nonlinear restoring force; Self-sustained oscillations; and Hill's equation and its application to the study of the stability of nonlinear oscillations. In addition there are six appendices where more mathematical aspects are considered. *N. Levinson.*

**Hazebroek, P., and van der Waerden, B. L. Theoretical considerations on the optimum adjustment of regulators.** Trans. A.S.M.E. **72**, 309–315 (1950).

The theory is founded on a least squares condition. *P. Franklin* (Cambridge, Mass.).

**Hazebroek, P., and van der Waerden, B. L. The optimum adjustment of regulators.** Trans. A.S.M.E. **72**, 317–322 (1950).

For regulators involving proportional and integral terms, a practical method of design based on the theory of the paper reviewed above is described. *P. Franklin.*

**Leonhard, A. Stabilitätskriterium insbesondere von Regelkreisen bei vorgeschriebener Stabilitätsgüte.** Arch. Elektrotechnik **39**, 100–107 (1948).

A method of testing whether the damping exceeds a given amount is described. It depends on transforming the roots of the characteristic equation. *P. Franklin.*

**Brock, John E. Some non-linear systems permitting simple harmonic motion.** J. Appl. Phys. **21**, 238–243 (1950).

Two nonlinear differential equations are constructed which have solutions expressible in terms of sine terms. *P. Franklin* (Cambridge, Mass.).

**Dubois-Violette, Pierre-Louis. Contribution à l'étude de la stabilité des circuits de régulation et des servomécanismes.** C. R. Acad. Sci. Paris **230**, 1380–1383 (1950).

Remarks on the extension of stability of a class of servomechanisms by the addition of a term depending on the rate. *P. Franklin* (Cambridge, Mass.).

**Dubois-Violette, Pierre-Louis. Sur la stabilité des régulateurs automatiques par action intégrale et dérivée seconde conjuguées.** C. R. Acad. Sci. Paris **230**, 1448–1450 (1950).

Remarks on the effect on stability of a class of servomechanisms of the addition of certain types of terms, including those involving an integral or second derivative. *P. Franklin* (Cambridge, Mass.).

**Dubois-Violette, Pierre-Louis. Étude de l'influence des temps de propagation sur la stabilité des servomécanismes régulateurs par la méthode de fusion des racines.** C. R. Acad. Sci. Paris **230**, 1499–1501 (1950).

Remarks on the stabilizing of servo-mechanisms which involve a time lag. *P. Franklin* (Cambridge, Mass.).

**Weyl, Hermann. Ramifications, old and new, of the eigenvalue problem.** Bull. Amer. Math. Soc. **56**, 115–139 (1950).

This Gibbs Lecture is a broad account of the ramifications of the eigenvalue problem, taking as a starting point the work of the author. The topics discussed include: the early work of the author on the Gibbs phenomenon for the heat equation and its connection with the equidistribution problem; the eigenfunctions and values of the second-order ordinary equation with a singular endpoint from the author's early work to the present; the asymptotic relations for the eigenfunctions and values of the vibrating membrane and the generalization of the problem to the case of a homogeneous manifold; integral equations and completeness relations on groups. *M. J. Gottlieb* (Chicago, Ill.).

**Szarski, J. Sur certaines inégalités entre les intégrales des équations différentielles aux dérivées partielles du premier ordre.** Ann. Soc. Polon. Math. **22** (1949), 1–34 (1950).

If $z_1$, $z_2$ are two solutions of the equation

$$p_1=f(x_1, \cdots, x_n; p_2, \cdots, p_n, z),$$

where $p_i=\partial z/\partial x_i$, this paper gives sufficient conditions that $z_1(a, x_2, \cdots, x_n)<z_2(a, x_2, \cdots, x_n)$ imply

$$z_1(x_1, x_2, \cdots, x_n)<z_2(x_1, x_2, \cdots, x_n)$$

for $x_1, \cdots, x_n$ in a certain set. It also gives generalizations for additional unknowns and additional equations. *J. M. Thomas* (Durham, N. C.).

**Satoh, Tunezo. A new analytical method for plane stress problems.** Jap. J. Math. **19**, 233–262 (1948).

The theory of Laplace's equation $\Delta w=w_{xx}+w_{yy}=0$, where $w(x, y)$ is a real-valued function, is related in a well-known manner to the theory of functions $f(x+iy)=u(x, y)+iv(x, y)$ of the complex variable $x+iy$ where $i^2+1=0$, $x$, $y$, $u(x, y)$, $v(x, y)$ are real and the functions $u$ and $v$ satisfy the Cauchy-Riemann equations $u_x=v_y$, $v_x=-u_y$. L. Sobrero [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 77–82, 135–140, 479–483 (1934)] has shown that the theory of the biharmonic equation $\Delta\Delta w=w_{xxxx}+2w_{xxyy}+w_{yyyy}=0$, where $w(x, y)$ is a real-valued function, is similarly related to the theory of functions

$$f(x+jy)=a(x, y)+jb(x, y)+j^2c(x, y)+j^3d(x, y)$$

of the hypercomplex variable $x+jy$ where $(j^2+1)^2=0$, $x$, $y$, $a(x, y)$, $b(x, y)$, $c(x, y)$, $d(x, y)$ are real, and the functions $a$, $b$, $c$, $d$ satisfy the Cauchy-Riemann equations $a_x=b_y$, $b_x=c_y-2a_y$, $c_x=d_y$, $d_x=-a_y$. Sobrero also showed that the theory of the biharmonic equation is related to the theory of analytic (in a certain sense) functions $f(\alpha+\bar{\omega}\beta)$ of the complex variable $\alpha+\bar{\omega}\beta$, where $\alpha$ and $\beta$ are ordinary complex numbers, and $\bar{\omega}^2=0$. Extending Sobrero's result, the present author shows that the theory of the equation

$$(*)\qquad \lambda_1^2\lambda_2^2w_{xxxx}+(\lambda_1^2+\lambda_2^2)w_{xxyy}+w_{yyyy}=0,$$

$\lambda_1$ and $\lambda_2$ being real constants, $w(x, y)$ real-valued, is related to the theory of functions

$$f(x+jy)=a(x, y)+jb(x, y)+j^2c(x, y)+j^3d(x, y)$$

Please see the note to p. 241, Tsuji.

where $j^4+(\lambda_1^2+\lambda_2^2)j^2+1=0$, $x, y, a(x, y)$, $b(x, y)$, $c(x, y)$, $d(x, y)$ are real, and the functions $a, b, c, d$ satisfy the Cauchy-Riemann equations $a_x=b_y$, $b_x=c_y-(\lambda_1^2+\lambda_2^2)(\lambda_1^2\lambda_2^2)^{-1}a_y$, $c_x=d_y$, $d_x=-(\lambda_1^2\lambda_2^2)^{-1}a_y$. It is also shown that the theory of equation (*) is related to the theory of analytic (in a certain sense) functions $f(\alpha+\bar{\omega}\beta)$, where $\alpha$ and $\beta$ are ordinary complex numbers, and $\bar{\omega}^2=\omega\bar{\omega}$, for a certain complex number $\omega$. All these considerations are closely related to those of Matsumoto [same vol., 441–482 (1947); these Rev. 10, 445] and the present paper seems to be the paper by Satoh referred to by that author. *J. B. Diaz.*

**Garnir, Henri. Les problèmes aux limites pour l'équation $\Delta u=k^2u$ dans une bande.** Revue Sci. 87, 33 (1949).

Given the equation $\Delta u=k^2u$, $k\neq 0$, in either two or three dimensions, the author notes that the solutions of the three classical boundary value problems for the region $0\leqq x\leqq d$ depend on Green's function for the region, and observes that the usual forms of Green's function for this region are meaningless if $k=0$. He gives new forms which yield significant results in the limiting case in which $k=0$.

*F. W. Perkins* (Hanover, N. H.).

**Wigner, E. P., and Eisenbud, L. A mathematical foundation of the resonance theory.** United States Atomic Energy Commission, Oak Ridge, Tenn., **MDDC 744**, 4 pp. (1947).

Gli autori studiano il problema di Neumann per l'equazione

$$(1)\qquad (H-\epsilon)\varphi=\left\{-\sum_{m=1}^{n}\frac{\partial^2}{\partial x_m^2}+V(x_1, \cdots, x_n)-\epsilon\right\}\varphi=0$$

con il seguente procedimento formale. Sia $I$ il dominio di integrazione per la (1), $S$ la frontiera, $\{F_k\}$ un sistema di funzioni ortonormale (e completo) su $S$. Cerchiamo un integrale $\varphi_k$ tale che sia, su $S$, (2) $\partial\varphi_k/\partial n=F_k$, dove $n$ è la normale esterna a $S$. Posto, su $S$, (3) $\varphi_s=\sum_t\rho_{st}(\epsilon)F_t$, si tratta di determinare le $\rho_{st}(\epsilon)$. Siano $E_i$ gli autovalori ed $X_i$ le autosoluzioni del problema di Neumann per la (1): (4) $(H-E_i)X_i=0$, $\partial X_i/\partial n=0$. Le $X_i$ costituiscono un sistema ortonormale completo in $I$. Si ha perciò, in $I$, (5) $\varphi_s=\sum_i\alpha_{si}(\epsilon)X_i$ e (per le formula di Green, le (1), (2), e (4))

$$(E_i-\epsilon)\int_I X_i\varphi_s dI=-\int_S\left\{\varphi_s\frac{\partial X_i}{\partial n}-X_i\frac{\partial\varphi_s}{\partial n}\right\}dS$$

$$=\int_S X_i\frac{\partial\varphi_s}{\partial n}dS=\int_S X_iF_s dS,$$

cioè, per le (5),

$$c_{is}=(E_i-\epsilon)\alpha_{si}(\epsilon)\quad (X_i=\textstyle\sum_s c_{is}F_s).$$

Si ha allora, dalla (5), supposta valida in $S$,

$$\sum_t\rho_{st}(\epsilon)F_t=\sum_i\frac{c_{is}}{E_i-\epsilon}X_i=\sum_{i,t}\frac{c_{is}c_{it}}{E_i-\epsilon}F_t,$$

da cui segue $\rho_{st}(\epsilon)=\sum_i c_{is}c_{it}/(E_i-\epsilon)$. *L. Amerio.*

**Ghizzetti, Aldo. Flow in a not homogeneous and anisotropic medium.** Ann. Soc. Polon. Math. 22 (1949), 195–200 (1950).

This paper concerns the uniqueness of solution of the general linear elliptic partial differential equation of second order in two variables and with constant coefficients, when the boundary conditions to be satisfied are that the normal derivative of the solution vanishes on the two infinite edges of a semi-infinite strip, the function itself assumes prescribed values upon the finite edge, and the function vanishes in a suitable fashion at $\infty$. The author concludes that there is at most one such solution of the equation if and only if certain inequalities are satisfied by the five constants occurring in the problem. The problem treated was proposed to the author by Litwiniszyn [cf. the same vol., 185–194 (1950); these Rev. 11, 699]. *C. Truesdell.*

**Germain, Paul, et Bader, Roger. Problème de Dirichlet pour une équation du type mixte.** C. R. Acad. Sci. Paris 230, 1824–1826 (1950).

The authors consider the equation

$$(*)\qquad k(y)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=F(x, y)$$

where $k(y)$ is positive when $y$ is positive and zero when $y=0$, and state they have proved there exists one and only one solution for this partial differential equation of the Dirichlet problem for a domain $D$ lying in the half-plane $y\geqq 0$ of which part of the boundary is a line segment lying on the axis $y=0$. The partial differential equation (*) is elliptic in that part of the domain for which $y>0$ and parabolic for $y=0$. This problem is similar to one that arises in gas dynamics for transonic flows. The method of proof is based on a metric formed by a complete orthonormal sequence of solutions of (*) introduced by Bergman for dealing with such problems. The same problem has been treated by Tricomi for the case where $k(y)=y$ and $F=0$.

*A. Gelbart* (Syracuse, N. Y.).

**Trjitzinsky, W. J. Singular elliptic-parabolic partial differential equations.** Ann. Soc. Polon. Math. 22, 43–96 (1949).

L'autore studia l'equazione ellittico-parabolica

$$(1)\qquad \sum_{i,j=1}^{m}a_{ij}(x, y)\frac{\partial^2u}{\partial x_i\partial x_j}+\sum_{i=1}^{m}b_i(x, y)\frac{\partial u}{\partial x_i}$$

$$+\sum_{i=m+1}^{n}b_i(x, y)\frac{\partial u}{\partial y_i}+c(x, y)u=f(x, y),$$

dove $x$ varie in un dominio limitato $B$ ed è $c_{ij}\leqq y_j\leqq c_{ij}$. Per $x$ appartenente alla frontiera $C$ di $B$ i coefficienti delle (1) e alcune derivate possono diventare infiniti. Utilizzando le formule di Green l'autore trasforma dapprima la (1) in una equazione integrale di prima specie di cui studia le chiusura. Successivamente, con convenienti ipotesi sulla natura dei coefficienti delle (1), trasforma tale equazione in una equazione regolare di seconda specie. *L. Amerio.*

**Erugin, N. P. A closed solution of a parabolic inhomogeneous boundary problem.** Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 215–217 (1950). (Russian)

The one-dimensional heat equation $u_t=u_{yy}$ with the boundary conditions $u(0, t)=$ constant, $u(l, t)=0$ $(t>0)$, $y(y, 0)=0$, $0<y<l$, is solved in terms of an improper integral. The author selects a certain improper integral which satisfies the differential equation and the first two boundary conditions and expands it for $t=0$ in a Fourier series in $y$ and determines the coefficients so that the remaining condition is satisfied. This solution is combined with the usual Fourier series solution to give the solution in closed form. The same process may be applied if the second boundary condition is modified to read $u(l, t)=$ constant.

*C. G. Maple* (Washington, D. C.).

**Del Pasqua, Dario. Risoluzione, con sole integrazioni, dell'equazione differenziale di tipo parabolico, con i dati di Cauchy su una curva assegnata.** Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 55–61 (1950).

The author gives an explicit solution in terms of quadratures of the Cauchy problem for the equation (1) $z_{xx}-az_y=0$, with the Cauchy data $z=\varphi_0(y)$, $z_x=\varphi_1(y)$ prescribed along a noncharacteristic curve $x=\psi(y)$. The solution is obtained by introducing the operator

$$If(x,y)=\int_{\psi(y)}^{x} f(t,y)\,dt.$$

Then (1) is reduced to the equation

(2) $$z_x'-aIz_y'=I\varphi_1(y)+aI\varphi_0(y)\psi'(y)+\varphi_0(y)$$

for the function $z'=Iz$. If $I$ is considered as a complex numerical parameter here, (2) becomes a first order equation with explicitly known general solution. Restoring its original operational meaning to $I$ in that solution leads to the desired explicit formula solving the Cauchy problem for (1). The inhomogeneous equation corresponding to (1) is solved similarly. *F. John* (Los Angeles, Calif.).

**Stampacchia, Guido. Il problema di Goursat per un'equazione alle derivate parziali del secondo ordine di tipo iperbolico.** Giorn. Mat. Battaglini (4) **3(79)**, 66–85 (1950).

The Goursat problem for a 2d order equation in 2 independent variables consists in determining a solution $u(x,y)$ with prescribed values on 2 curves $\Gamma_1$ and $\Gamma_2$ in the $(x,y)$-plane. The problem has been solved "in the small" by Cinquini-Cibrario [Ann. Mat. Pura Appl. (4) **21**, 189–229 (1942); these Rev. **6**, 4] for the most general hyperbolic equation $F(x,y,u,p,q,r,s,t)=0$, where in the customary notation $p, q, r, s, t$ denote the first and second partial derivatives of $u$. The present paper deals with the problem "in the large" for an equation of the special form $s=f(x,y,u,p,q)$ with characteristics parallel to the coordinate axes. Assume $\Gamma_1$ and $\Gamma_2$ to intersect at a point $O$ and to have tangents at $O$ not divided harmonically by the coordinate axes. Also, $\Gamma_1$ shall have no vertical tangent, $\Gamma_2$ no horizontal tangent, and $\Gamma_2$ shall not be separated from the $y$-axis by $\Gamma_1$ or by the curve symmetric to $\Gamma_1$ with respect to the $y$-axis. The rectangle $R$ bounded by the vertical lines through the endpoints of $\Gamma_1$ and the horizontal lines through the endpoints of $\Gamma_2$ shall contain $\Gamma_1$ and $\Gamma_2$. Then, under certain regularity conditions on $\Gamma_1$, $\Gamma_2$, $f$ and the data, a solution $u$ of the Goursat problem exists and is uniquely determined in $R$. *F. John.*

**Elianu, I. P. Le problème de Cauchy pour l'équation des ondes itérée.** Bull. Math. Soc. Roumaine Sci. **48**, 102–144 (1947).

The author studies the iterated wave equation. Proceeding in a way analogous to that used in the theory of polyharmonic functions he obtains the fundamental solution and extends Green's formula. Then using the idea of "finite part of an infinite integral" of Hadamard and some results of Nicolesco, he obtains the solution of Cauchy's problem. *L. Amerio* (Milan).

**Conti, Roberto. Sul problema di Cauchy per le equazioni di tipo misto $y^kz_{xx}-x^kz_{yy}=0$.** Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 105–130 (1950).

The equation $y^kz_{xx}-x^kz_{yy}=0$ is considered for positive integers $k$. The equation is hyperbolic for $xy>0$, parabolic for $xy=0$, and elliptic or hyperbolic for $xy<0$ according to the parity of $k$. The Cauchy data $z$ and $z_y$ are prescribed along the parabolic interval $0\leqq x_0<x<x_1$, $y=0$. It is proved that for Cauchy data of class $C^2$ a solution exists and is uniquely determined in the "characteristic domain" of the initial interval. This domain for odd $k$ is the triangular convex domain $T$ bounded by the initial interval and the characteristics through its endpoints. For even $k$ the characteristic domain consists of $T$ and its mirror image with respect to the $x$-axis. The case where the initial interval contains the origin in its interior is to be discussed elsewhere. *F. John* (Los Angeles, Calif.).

**Conti, Roberto. Sul problema di Cauchy per l'equazione di tipo misto $x^2z_{xx}-y^2z_{yy}=0$.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 577–582 (1949).

The following Cauchy problem is solved for the equation $x^2z_{xx}-y^2z_{yy}=0$. Let $C$ be a curve given by an equation $y=f(x)$ for $0\leqq x\leqq 1$, such that $C$ does not intersect any characteristic in more than one point, and such that $f>0$, $f(1)=1$. Along $C$ the values of $z_x$ and $z_y$ are prescribed, in addition to the value of $z$ at one point of $C$. By Riemann's method a solution of this Cauchy problem can be constructed in the region: $0\leqq x\leqq 1$, $x<y$, $xy<1$. If the "qualitative" condition is added, that the solution is regular at the origin, a functional equation has to be satisfied by the Cauchy data on $C$. *F. John* (Los Angeles, Calif.).

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## *Functional Analysis, Ergodic Theory*

**Dieudonné, Jean, et Gomes, Alfredo Pereira. Sur certains espaces vectoriels topologiques.** C. R. Acad. Sci. Paris **230**, 1129–1130 (1950).

To extend results of Köthe [Math. Z. **51**, 317–345 (1948); these Rev. **10**, 255], let $\{b_{mn}\}$ be a matrix of positive terms, let $r$ be $\geqq 1$, and let $E$ be the space of all infinite sequences $u=\{u_n\}$ such that $p_m(u)=(\sum_{n<\infty} b_{mn}|u_n|^r)^{1/r}<\infty$ simultaneously. Then $E$ is called the enterraced space of order $r$ (with Köthe $r=1$). (To get examples different from the ordinary $l^r$ sequence spaces, the authors impose the conditions $1\leqq b_{mn}\leqq b_{m+1,n}$ and $\lim_n b_{m+1,n}/b_{mn}=\infty$.) For $r>1$, the dual or conjugate space $E'$ of $E$ is the space of all sequences $\{x_n\}$ such that $\sum_n b_{mn}^{1/r}|x_n|^s<\infty$ for at least one $m$, where $s=r/(r-1)$; then $E$ is reflexive. Call a linear topological space $E$ of type $(M)$ if $E$ is locally convex, metrizable, complete, and if every bounded set is conditionally compact. It is asserted that the following conditions on an enterraced space $E$ of order $r\geqq 1$ are equivalent: (1) $E$ is of type $(M)$; (2) for each sequence $n_k\to\infty$ there exists $m$ such that $\lim_k b_{mn_k}=+\infty$; (3) every sequence weakly convergent in $E'$ is strongly convergent. *M. M. Day* (Urbana, Ill.).

**Everett, C. J., and Ryser, H. J. Rational vector spaces. II.** Duke Math. J. **17**, 135–145 (1950).

This paper studies algebraic number fields from the point of view of the general theory of rational vector spaces developed previously [Duke Math. J. **16**, 553–570 (1949); these Rev. **11**, 368]. Let $R(\eta)$ be a normal algebraic number field with the property that the operation $\alpha^*$ of taking the complex conjugate commutes with every automorphism of $R(\eta)$. It is shown that such a field is a rational inner-product space with the inner product $(\alpha,\beta)=T(\alpha^*\beta)$, where $T(\alpha)$ denotes the trace of $\alpha$. Fields with an orthonormal basis

with respect to this inner product are called $I$-fields. The authors investigate these $I$-fields and show in particular that whereas there is only one $I$-field of degree two, namely $R(i)$, every normal cubic field is an $I$-field, and there exist infinitely many $I$-fields of degree $n$ where $n>2$. The final theorem of the paper characterizes the completion of $R(\eta)$ with respect to the pseudo-evaluation $\|\alpha\|=\sqrt{T(\alpha^*\alpha)}$, with the aid of some results of K. Mahler [Acta Math. 66, 79–119 (1936)]. *D. H. Hyers* (Los Angeles, Calif.).

**Favard, J. Sur l'approximation dans les espaces vectoriels.** Ann. Mat. Pura Appl. (4) **29**, 259–291 (1949).

This paper is primarily expository. The author sketches a general approach to problems of approximation of functions by considering abstract spaces. More attention is given to the presentation of general ideas than to individual results. *A. Zygmund* (Cambridge, Mass.).

**Taylor, Angus E. On certain Banach spaces whose elements are analytic functions.** Actas Acad. Ci. Lima **12**, 31–43 (1949).

The Banach space $H^p$, $p>1$, is defined to be the space of complex functions analytic in the circle $|z|<1$ and for which

$$\|f\| = \sup_{0\leq r<1}\left\{(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right\}^{1/p} < \infty.$$

The space $L^p$ is the space of complex functions defined and measurable on the circle $C$, $|z|=1$, with

$$\|f\| = \left\{(2\pi)^{-1}\int_0^{2\pi}|f(e^{i\theta})|^p d\theta\right\}^{1/p} < \infty.$$

It is shown that $H^p$ is isometric to the subspace $M$ of functions of $L^p$ whose negatively indexed Fourier coefficients vanish and that the general linear functional over $H^p$ is given by $W(f)=(2\pi i)^{-1}\int_C t^{-1}f(t)F(t)dt$ where $F\epsilon H^q$, $1/q+1/p=1$. *R. E. Fullerton* (Madison, Wis.).

**Lorch, E. R. Return to the self-adjoint transformation.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 137–144 (1950).

The author presents a proof of the spectral decomposition for an unbounded self-adjoint transformation $H$ in a Hilbert space. The argument rests on the consideration of the integral

$$K=(2\pi i)^{-1}\int_C(\zeta-\lambda)(\mu-\zeta)(\zeta I-H)^{-1}d\zeta$$

(where $C$ denotes a path in the complex plane passing by the points $\lambda$ and $\mu$, $\lambda<\mu$, of the real axis) and on the observation that $H$ is bounded and self-adjoint on the range of the transformation $K$ where it satisfies the inequality $\lambda I\leqq H\leqq\mu I$. *L. Nachbin* (Rio de Janeiro).

**Julia, Gaston. Quelques caractères des opérateurs hermitiens permutables ou antipermutables.** Ann. Mat. Pura Appl. (4) **30**, 185–189 (1949).

The content of this paper is essentially the same as that of another paper already reviewed [C. R. Acad. Sci. Paris **228**, 962–964 (1949); these Rev. **10**, 547]. *P. R. Halmos* (Chicago, Ill.).

**Livšic, M. S. Isometric operators with equal deficiency indices, quasi-unitary operators.** Mat. Sbornik N.S. **26(68)**, 247–264 (1950). (Russian)

On a complex Hilbert space $H$ let $V$ be a partially isometric operator with initial space $G$ and final space $G'$, such that $\dim(H\ominus G)=\dim(H\ominus G')=m<\infty$. If $\{g_k: k=1,\cdots,m\}$ and $\{g_k': k=1,\cdots,m\}$ are orthonormal bases in $H\ominus G$ and $H\ominus G'$, respectively, and if $z$ is a complex number with $|z|<1$, write

$$w_{kj}(z)=((V-z)(1-zV^*)^{-1}g_k, g_j'), \qquad k, j=1,\cdots,m.$$

The matrix function $w=(w_{kj})$ is called the characteristic function of $V$. The author's two principal results may be stated as follows. (I) A necessary and sufficient condition that a matrix function be the characteristic function of some $V$ is that $w$ be analytic, have norm less than 1, and vanish at the origin. (II) Two partially isometric operators, neither of which is unitary on any subspace of $H$, are unitarily equivalent if and only if there exist two $m\times m$ unitary matrices $u'$ and $u''$ such that the characteristic functions $w_1$ and $w_2$ satisfy $w_1=u'w_2u''$. *P. R. Halmos.*

**Livšic, M. S., and Potapov, V. P. A theorem on the multiplication of characteristic matrix functions.** Doklady Akad. Nauk SSSR (N.S.) **72**, 625–628 (1950). (Russian)

The authors state (without proof) five theorems of which the principal one exhibits a complete set of invariants for the unitary equivalence of two quasi-unitary operators. The distinguishing invariant is a suitably normalized characteristic matrix function associated with such operators. For the terminology see the preceding review. *P. R. Halmos* (Chicago, Ill.).

**Lukomskiĭ, T. I. On the theory of matrix representations of unbounded self-adjoint operators.** Doklady Akad. Nauk SSSR (N.S.) **70**, 377–379 (1950). (Russian)

A matrix representation of an operator $A$ on a separable Hilbert space $H$ is a pair $\{\{e_k\},\{a_{ik}\}\}$, where $\{e_k\}$ is an orthonormal basis of $H$ and $a_{ik}=(Ae_k, e_i)$. If $A$ is a closed operator, a basis $\{e_k\}$ is $A$-admissible if $A$ is the closure of its restriction to the linear manifold spanned by $\{e_k\}$. The symbol $\mathfrak{U}_A$ denotes the set of all unitary operators which map the domain of $A$ onto itself. The author states (without proof) four theorems concerning admissible bases; the following two are typical. Theorem 3. A necessary and sufficient condition that a finite set of closed operators have a common admissible basis is that each of them be the closure of its restriction to the intersection of their domains. Theorem 4. A necessary and sufficient condition that both the unitary operators $U$ and $U^*$ send every $A$-admissible basis on an $A$-admissible basis is that $U$ belong to $\mathfrak{U}_A$. The paper concludes with a definition of unitary equivalence for two matrix representations $\{\{e_k\},\{a_{ik}\}\}$ and $\{\{e_k'\},\{a_{ik}'\}\}$ of a symmetric operator $A$; the definition requires that $a_{ik}'=\sum_{pq}u_{pi}(\sum_{rs}u_{qk}a_{pq})=\sum_{rs}u_{qk}(\sum_{pq}u_{pi}a_{pq})$, where $u_{pq}=(e_q', e_p)$, and that the unitary operator defined by the matrix representation $\{\{e_k\},\{u_{ik}\}\}$ belong to $\mathfrak{U}_A$. *P. R. Halmos.*

**Schmeidler, Werner. Über unbeschränkt iterierbare Operatoren.** J. Reine Angew. Math. **187**, 109–114 (1949).

Two theorems on the representability by Jacobi-matrices of linear operators between Hilbert spaces, where the operators have to satisfy a strong condition involving iteration. The author does not seem to be aware that his second theorem is contained in M. H. Stone's Linear Transformations in Hilbert Space and Their Applications to Analysis [Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932, theorem 7.17], where it is proved under much less restrictive conditions. *F. I. Mautner.*

**Krein, M. Infinite $J$-matrices and a matrix-moment problem.** Doklady Akad. Nauk SSSR (N.S.) **69**, 125–128 (1949). (Russian)

A regular $J_p$-matrix (a generalization of the Jacobi-matrix) is defined as an infinite Hermitian matrix $\{a_{\alpha,\beta}\}$ $(\alpha, \beta = 0, 1, \cdots)$ having the form $\{A_{i,k}\}$ $(i, k = 0, 1, \cdots)$, where $A_{i,k}$ are $p$-matrices (i.e., with $p$ rows and $p$ columns), and where $A_{i,k}$ is the null-matrix 0 for $|i-k|>1$ and $A_{i,i+1}$ is not singular. A $J_p$-matrix is related to a matrix-moment problem

$$S_n = \int_{-\infty}^{\infty} \lambda^n dT(\lambda), \qquad n = 0, 1, \cdots, \tag{1}$$

where $S_n$ are given $p$-matrices and $T(\lambda)$ is to be found. The corresponding theory has the features of the classical theory concerning ordinary Jacobi-matrices $J_1$. Consider the class $\mathfrak{P}$ of all polynomials $P(\lambda) = C_0 + C_1\lambda + \cdots + C_n\lambda^n$ $(n \geqq 0)$ with the complex variable $\lambda$ and with $C_i$ belonging to the collection $\mathfrak{M}_p$ of all $p$-matrices with complex elements. Define $P^*(\lambda)$ by the relation $P^*(\bar{\lambda}) = [P(\lambda)]^*$. To a given $J_p$-matrix $A$ there are attached polynomials $D_k(\lambda) \in \mathfrak{P}$ $(k = 0, 1, 2, \cdots)$ defined by the recurrent equations

$$A_{k,k-1}D_{k-1}(\lambda) + (A_{k,k} - \lambda I)D_k(\lambda) + A_{k,k+1}D_{k+1}(\lambda) = 0$$

$(k = 0, 1, \cdots)$, where $D_{-1} = 0$ and $D_0$ is an arbitrarily chosen nonsingular constant $p$-matrix [cf., e.g., M. H. Stone, Linear Transformations in Hilbert Space . . . , Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932, p. 546]. The limit $H(z) = \lim_{n\to\infty} (\sum_{k=0}^{n} D_k^*(\bar{z}), D_k(z))^{-1}$ exists for every complex $z$, and the rank $r(z)$ of the Hermitian matrix $H(z)$ is the same for all $z$ with $\Im(z) > 0$, and analogously for $\Im(z) < 0$ [H. Nagel, Math. Ann. **112**, 247–285 (1936)]. These ranks $\nu_+$, $\nu_-$ are identical with the deficiency indices of the Hermitian operator corresponding to the matrix $A$. For two polynomials $P(\lambda) = \sum_{k=0}^{m} U_k D_k(\lambda)$, $Q(\lambda) = \sum_{k=0}^{m} V_k D_k(\lambda)$ with constant matrices $U_k$, $V_k$ the "scalar product" is defined by $\{P, Q\} = \sum_{k=0}^{s} U_k V_k^*$, $s = \min[m, n]$. We have $\{D_i, D_k\} = \delta_{ik} I$, the scalar product is distributive, and $\{CP, Q\} = C\{P, Q\}$, $\{P, CQ\} = \{P, Q\}C^*$. If we put (2) $S_n = \{\lambda^n I, I\}$ $(n = 0, 1, \cdots)$, we have

$$\sum_{j,k=0}^{n} X_j S_{j+k} X_k^* = \left\{ \sum_{j=0}^{n} X_j \lambda^j, \sum_{j=0}^{n} X_j \lambda^j \right\}$$

for any $X_j \in \mathfrak{M}_p$ $(j = 0, 1, \cdots)$, and we have

$$\sum_{j,k=0}^{n} x_j S_{j+k} x_k^* > 0 \quad (n = 0, 1, \cdots) \tag{3}$$

for any vectors $x_j = (\xi_{j1}, \cdots, \xi_{jp})$. Conversely, any sequence $S_n \in \mathfrak{M}_p$ satisfying (3) can be defined by (2) for some properly chosen $J_p$-matrix $A$. The condition (3) is necessary and sufficient for the existence of a solution of the matrix-moment problem (1) (with $xT(\lambda)x^*$ nondecreasing in $(-\infty, \infty)$ and with an infinite number of points of increase). This problem has a unique "normed" solution (i.e., $\lim_{\lambda\to-\infty} T(\lambda) = 0$, $T(\lambda - 0) = T(\lambda)$, $-\infty < \lambda < \infty$) if and only if either $\nu_+$ or $\nu_-$ equals 0. The case of the undetermined matrix-moment problems is considered too, and theorems corresponding to those of the classical theory of the Hamburger moment problem are stated. There are no proofs. The author refers to work in a paper unavailable outside the USSR and to Krasnoselskiĭ and Kreĭn [Uspehi Matem. Nauk **2**, no. 3(19), 60–106 (1947); these Rev. **10**, 198]. *O. M. Nikodým.*

**Nikodym, Otton Martin. Un nouvel appareil mathématique pour la théorie des quanta.** Ann. Inst. H. Poincaré **11**, 49–112 (1949).

The main purpose of this paper is to present a theory adequate for the treatment of spectral multiplicities of Hermitian operators on a separable Hilbert space. The objects of principal interest are Boolean algebras of projections. Though all the essential results are known, the author's point of view is different from that of (say) Wecken or Nakano. A novel technical aspect of the work is the consideration of chains (totally ordered sets) of projections generating a given Boolean algebra; roughly speaking, the completion by cuts of such a chain (with a measure induced by a suitable vector in Hilbert space) can be used as a measure space whose $L_2$ is in a natural isomorphism with the Hilbert space, and which puts into evidence the structure of the Boolean algebra. When applied to (not necessarily bounded) self-adjoint operators, the theory yields the known fact that every such operator may be viewed as a multiplication by a measurable function on a suitable concrete Hilbert space. *P. R. Halmos* (Chicago, Ill.).

**Citlanadze, Ê. S. On the variational theory of a class of nonlinear operators in the space $L_p$ $(p>1)$.** Doklady Akad. Nauk SSSR (N.S.) **71**, 441–444 (1950). (Russian).

The results of earlier articles [e.g., same Doklady (N.S.) **57**, 879–881 (1947); these Rev. **9**, 447] are extended to $L^p$ spaces. Let $f(x)$ be a weakly continuous functional of $x(s)$ in $L^p(0, 1)$, satisfying $f(x) \geqq 0$, $f(x) = 0$ if and only if $x = 0$, $f(x)$ with everywhere Fréchet differentiable. For any $x$ and $h$ in $L^p$ the Fréchet differential $df(x; h)$ is a linear functional in $h$, and hence is of the form $df(x; h) = (Lx, h)$, where $Lx$ is an element of the conjugate space $L^q$ and $L$ is a (generally nonlinear) function on $L^p$ to $L^q$, such that $Lx = 0$ only if $x = 0$. It is assumed that $Lx$ is Fréchet differentiable, and satisfies a Lipschitz condition $\|Lx - Lx'\| \leqq M\|x - x'\|$ inside the unit sphere $S_1$ of $L^p$. Among results stated without proof are the following. If $Lx$ is generated by a weakly continuous $f(x)$ which satisfies $\|f(x+h) - f(x) - (Lx, h)\| \leqq c\|h\|^2$, then $L$ is totally continuous. An element $x$ of $S_1$, on the surface of $S_1$, is called a normal characteristic element of $Lx$ for the characteristic value $\lambda$ if $Lx(s) = \lambda |x(s)|^{p/q} \operatorname{sgn} x(s)$ almost everywhere on $(0, 1)$.

Let $S_1^*$ be the space obtained by identifying points $x$ and $-x$ in $S_1$. Let $[P^*]_k$ denote the set of all closed compact sets in $S_1^*$ whose category, in the sense of Lusternik and Schnirelmann, is greater than or equal to $k$. Let $c$ be the upper bound of the minima of $f(x)$ on all sets of $[P^*]_k$. Sets of all possible categories exist in $S_1^*$, and it is assumed that $f(x)$ is even so that it defines a function in $S_1^*$. Then the operator $Lx$ has at least one characteristic element on each level surface $(f = c)$. *J. L. B. Cooper* (Cardiff).

**Rickart, C. E. The uniqueness of norm problem in Banach algebras.** Ann. of Math. (2) **51**, 615–628 (1950).

A Banach space (B.S.) or Banach algebra (B.A.) is said to have a unique norm (u.n.) if any two norms relative to which it is a B.S. (B.A.) give rise to equivalent topologies. This paper is a systematic study of semisimple (s.s.) B.A.'s with u.n. which include among many new ones Gelfand's theorem on s.s. commutative complex B.A.'s [Rec. Math. [Mat. Sbornik] N.S. **9**(**51**), 3–24 (1941), theorem 17; these Rev. **3**, 51]; and Eidelheit's theorem on the B. A. of all bounded operators on a real B.S. [Studia Math. **9**, 97–105 (1940), theorem 1; these Rev. **2**, 224; **3**, 51; **7**, 620]. The B.S.'s considered here are over the reals unless otherwise

stated; the B.A.'s satisfy the condition $\|xy\| \leqq \|x\| \cdot \|y\|$; a unit, or commutativity, is not assumed. The principal results are the following. (1) A strongly s.s. [see I. E. Segal, Trans. Amer. Math. Soc. **61**, 69–105 (1947); these Rev. **8**, 438] B.A. has u.n. (2) A B.A. with a family of minimal left ideals whose left annihilators have intersection (0) has u.n. (Such B.A.'s are also s.s. in the sense of Jacobson.) This implies that a primitive B.A. with minimal ideals has u.n., and also that an irreducible B.A. of linear operators over a linear vector space containing nonzero operators with finite dimensional range has u.n. (3) An $A^*$-B.A. is a B.A. with an anticommutative involution $x^*$ which has, in addition to its norm, an "auxiliary norm" satisfying the usual requirements, and, in addition, $|xy| \leqq |x|\,|y|$ and $|x|^2 \leqq k|x^*x|$ where $k$ is a positive constant (completeness with respect to the auxiliary norm is not assumed). Examples of such B.A.'s are s.s. commutative B.A.'s with the identity involution, *-closed B.A.'s of bounded operators on Hilbert space. $A^*$-B.A.'s are s.s. and * is continuous. An $A^*$-B.A. has u.n. (4) Several "reduction theorems" are proved. The u.n. problem for a s.s. B.A. $B$ can be reduced to that for primitive ones (i.e., for all primitive quotient B.A.'s $B-P$ where $P$ is a primitive ideal of the form $(M:B)$ where $M$ is a maximal regular right ideal of $B$; see Jacobson [Amer. J. Math. **67**, 300–320 (1945); Ann. of Math. (2) **48**, 8–21 (1947); these Rev. **7**, 2; **8**, 433]). Primitive real (complex) s.s. B.A.'s being isomorphic to irreducible (dense in the sense of Jacobson) algebras of bounded linear transformations on a real (complex) B.S., the u.n. problem for a real (complex) s.s. B.A. can thus be reduced to that for irreducible (dense) primitive B.A.'s of bounded operators on a real (complex) B.S. (5) A homomorphism $f$ of an arbitrary B.A. either onto a s.s. B.A. with u.n. or into an $A^*$-B.A. such that $(f(B))^* = f(B)$ (e.g., into a s.s. commutative B.A.) is continuous. Among other things, the proofs utilise strongly various properties of the set $S$ defined for a B.A. $B$ with two norms as follows: $s$ is in $S$ if and only if there exist $x_n$ in $B$ such that $\|x_n - s\| \to 0$ for one norm and $\|x_n\|_1 \to 0$ for the other norm. The set $S$ is an ideal of $B$ closed in both topologies and u.n. is equivalent to $S=(0)$. Thus, for instance, simple B.A.'s have u.n., and u.n. in the quotient space $B-I$ ($I$ a closed ideal) implies that $S \subseteq I$. *G. K. Kalisch.*

**Dowker, Yael Naim. A new proof of the general ergodic theorem.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 162–166 (1950).

The ergodic theorem of Hurewicz, in the form given to it by Halmos [Proc. Nat. Acad. Sci. U. S. A. **32**, 156–161 (1946); these Rev. **8**. 34], is generalized so as to apply to a measurable transformation that is single-valued but not necessarily one-to-one. The method of proof is similar to that used by F. Riesz [Comment. Math. Helv. **17**, 221–239 (1945); these Rev. **7**, 255] to obtain a corresponding generalization of Birkhoff's theorem. *J. C. Oxtoby.*

**Êl'sgol'c, L. Ê. An estimate for the number of singular points of a dynamical system defined on a manifold.** Mat. Sbornik N.S. **26(68)**, 215–223 (1950). (Russian)

Let $M^2$ be a compact two-dimensional manifold and $dx_i/dt = X_i(x_1, x_2)$, $i=1, 2$, a dynamical system over $M^2$, where the $X_i$ are continuous in the local coordinates $x_i$ and are such that the problem of Cauchy has a unique solution over $M^2$. It is also assumed that there are only a finite number of singular points (points where $X_1 = X_2 = 0$). The algebraic number of the singular points (sum of their indices) is $\chi(M^2)$, the Euler characteristic. When the vector $(X_1, X_2)$ is a gradient more accurate information has been given for both the algebraic and actual number of the singular points, and this even for a compact $M^n$ [M. Morse, The Calculus of Variations in the Large, Amer. Math. Soc. Colloquium Publ., v. 18, New York, 1934; Lusternik and Schnirelmann, Méthodes topologiques dans les problèmes variationnels I, Actualités Sci. Ind., no. 188, Hermann, Paris, 1934; Froloff and the author, Rec. Math. [Mat. Sbornik] (1) **42**, 637–642 (1935)]. The author gives a considerable extension of these results on the assumption that the $\alpha$- and $\omega$-sets of the system are made up of the simplest possible singular points in finite number: stable or unstable foci or nodes and saddle points. The general method consists in surrounding each unstable node or focus with a Jordan curve without contact. Assuming that the curves are all crossed outwards at time $t_0$, as $t$ increases from $t_0$ and the paths are followed there results an isotopic deformation of the neighborhoods of the unstable points in question until there is merely left an arbitrarily small neighborhood of the stable foci or nodes and of the saddle points with the trajectories departing from them. The resulting decomposition of $M^2$ yields the inequalities of the paper. As an application, if $M^2$ is a projective plane or a torus there are at least three distinct singular points. The same method is applied to the case where besides the singular points (still of the same simple type) there is a stable or unstable limit-cycle. Under similar assumptions an extension is made to any $M^n$ under the restriction that no trajectory joins two generalized saddle points (points which are conditionally stable).

*S. Lefschetz* (Princeton, N. J.).

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## *Theory of Probability*

**Ghosh, M. N. Expected travel among random points in a region.** Calcutta Statist. Assoc. Bull. **2**, 83–87 (1949).

The author considers the problem of obtaining the expected length of the shortest path joining a system of $n$ random points in a plane area. He shows (somewhat unrigorously) that a particular choice of path, obtained by dividing the area into small squares, joining randomly chosen points in one square, proceeding to the next, and so on, yields an expected length not very different from the expected shortest length found by Marks [Ann. Math. Statistics **19**, 419–422 (1948); these Rev. **10**, 131]. The author therefore suggests that in practical survey problems his path is an acceptable substitute for the shortest one.

*J. L. Doob* (Urbana, Ill.).

**Bartlett, M. S. Recurrence times.** Nature **165**, 727–728 (1950).

The author points out a slip in Chandrasekhar's survey [Rev. Modern Physics **15**, 1–89 (1943); these Rev. **4**, 248]. Lumping together several states of a Markov process spoils the Markovian character, and hence a certain calculation of a recurrence time. The author shows how at least the average recurrence time to a group of states can be calculated. *W. Feller* (Princeton, N. J.).

**Dvoretzky, A., Erdös, P., and Kakutani, S. Double points of paths of Brownian motion in $n$-space.** Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 75–81 (1950).

The following assertions relate to an $n$-dimensional Brownian motion. According to Lévy [Amer. J. Math. **62**,

487–550 (1940); these Rev. **2**, 107] if $n=2$ almost all paths have double points. According to Kakutani [Proc. Imp. Acad. Tokyo **20**, 648–652 (1944); these Rev. **7**, 315] if $n \geqq 5$ almost no paths have double points. It is proved that the Lévy result holds also when $n=3$, and that the Kakutani result holds when $n=4$. The proof depends on the following two lemmas. When $n=3$ almost all paths (finite parameter interval) have positive 3-dimensional capacity; when $n=4$ almost all paths (finite parameter interval) have zero 4-dimensional capacity. *J. L. Doob* (Urbana, Ill.).

**Moran, P. A. P. The oscillatory behaviour of moving averages.** Proc. Cambridge Philos. Soc. **46**, 272–280 (1950).

The author analyzes the effect of the iteration of a moving average on an orthonormal sequence of random variables. He finds that finite stretches of the new process are, after a large number of iterations, asymptotically very nearly the sum of a constant and a finite number of sine waves. That is, the normalized spectral distribution of the new process is very nearly one confined to a finite number of points. The basic hypotheses are that the average is a finite average involving at least two terms, but the method applies to infinite averages, and the same type of result is obtained if the original random variables are only supposed to form a stationary sequence with a spectral density which does not vanish at a finite point set determined by the moving average. *J. L. Doob* (Urbana, Ill.).

**Ghidoli, Gian Bruno. Generalizzazione della legge sinusoidale.** Statistica, Milano **9**, 72–82 (1949).

The author is apparently unaware of the work of V. Romanovsky [Rend. Circ. Mat. Palermo **56**, 82–111 (1932); **57**, 130–136 (1933)]. *H. Wold* (Uppsala).

**Moyal, J. E. Stochastic processes and statistical physics.** J. Roy. Statist. Soc. Ser. B. **11**, 150–210 (1949).

**Bartlett, M. S. Some evolutionary stochastic processes.** J. Roy. Statist. Soc. Ser. B. **11**, 211–229 (1949).

**Kendall, David G. Stochastic processes and population growth.** J. Roy. Statist. Soc. Ser. B. **11**, 230–264 (1949).

Expository papers, with valuable bibliographies and historical remarks. The mathematics is mostly formal in nature, and detailed proofs are omitted. The authors derive the results from their own points of view, frequently giving new insight into the problems considered. Of particular interest, for example, is the systematic operational method used by Moyal and Bartlett of deriving differential equations for probability distributions in stochastic processes (Fokker-Planck equations and others). Moyal's paper stresses detailed applications to statistical physics. Bartlett's stresses random walks and multiplicative (birth) processes from a general point of view. Kendall is interested in the details of birth and death processes from the point of view of constructing models for various applications. He obtains new results on the stochastic fluctuations of the age distributions of a population, assuming a continuous time parameter. *J. L. Doob* (Urbana, Ill.).

***Symposium on Stochastic Processes.** Royal Statistical Society, London, 1949. 133 pp. 12s/6d.

Reprint of the three papers reviewed above and of discussions of them.

**Bode, H. W., and Shannon, C. E. A simplified derivation of linear least square smoothing and prediction theory.** Proc. I.R.E. **38**, 417–425 (1950).

The authors present a (nonrigorous) treatment of the smoothing and prediction theory of stationary stochastic processes from the point of view of electrical circuit theory. *J. L. Doob* (Urbana, Ill.).

**Pollaczek, F. Réduction de différents problèmes concernant la probabilité d'attente au téléphone, à la résolution de systèmes d'équations intégrales.** Ann. Inst. H. Poincaré **11**, 135–173 (1949).

[This is a continuation of the paper in the same Ann. **10**, 1–55 (1946); these Rev. **9**, 362.] First the author derives the distribution of waiting times under various extremely general conditions. In each case one arrives at an exceedingly complicated system of linear integral equations related to the Cauchy type and with linear side conditions. The left sides are known and vary from case to case. The solution of the integral equations is given using a special technique introduced for that purpose in a companion paper [same vol., 113–133 (1949); these Rev. **11**, 660]. *W. Feller* (Princeton, N. J.).

**Jacobæus, Christian. A study on congestion in link systems.** Ericsson Technics no. **48**, 68 pp. (1950).

In telephone exchanges various groups of devices are used in series and congestion or blocking can occur in each of them. In conventional theories each group is treated separately and various methods are used for different types of connections. The author unifies the theory considerably by defining, in a very general way, link systems which include all sorts of arrangements from manual exchanges to most complicated systems. A typical link connection system consists of $m^2$ input channels, $n^2$ output channels, and $mn$ link devices, each group being arranged in a matrix. The $m$ input channels of the $j$th column have access to the $n$ link devices of the corresponding column and each of these in turn has full access to the $n$ output channels of the corresponding row. The author treats the congestion problem at length, but the details are too technical for a short review. *W. Feller* (Princeton, N. J.).

**Faragó, P. S., and Takács, L. The probability distribution of the number of secondary electrons.** Hungarica Acta Physica **1**, no. 6, 43–52 (1949).

Each primary electron has the probability $p_k$ of emitting $k$ secondary electrons, and the process is of the chain reaction type. This paper contains some theoretical considerations on the method of an experimental determination of the distribution $\{p_k\}$. The estimates are based on the method of moments. *W. Feller* (Princeton, N. J.).

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## *Mathematical Statistics*

**Barnard, G. A. Statistical inference.** J. Roy. Statist. Soc. Ser. B. **11**, 115–139; discussion, 139–149 (1949).

An exposition of the author's theory of statistical inference, with critical comments by others. The questions involved are in large part philosophic. There seem to be points of agreement with R. A. Fisher's theory of fiducial probability. The likelihood ratio plays an important role in the theory. *J. Wolfowitz* (New York, N. Y.).

**Castellani, Maria. On multinomial distributions with limited freedom: A stochastic genesis of Pareto's and Pearson's curves.** Ann. Math. Statistics 21, 289–293 (1950).

A quasi-mechanical model similar to that used in classical statistical mechanics is set up, which, when the probability of the system being in a given configuration is maximized, leads to a distribution which includes the Pearson and Pareto curves. Then an attempt to explain the success of these curves in the study of certain social phenomena is made. Both the physical arguments and the mathematical reasoning seem to be intended to be suggestive rather than strictly cogent. *B. O. Koopman* (New York, N. Y.).

**Quenouille, M. H. The evaluation of probabilities in a normal multivariate distribution, with special reference to the correlation ratio.** Proc. Edinburgh Math. Soc. (2) 8, 95–100 (1949).

Let $f(x_1, x_2, \cdots, x_n) = (2\pi\sigma^2)^{-n/2} \exp(-(2\sigma^2)^{-1}\sum x_i^2)$, the multivariate normal distribution for a series of independent observations with common variances $\sigma^2$, or for the firing of shots at a target. The author is interested in finding the probability of falling within a region at a given distance from the point $P$ given by $(a_1, a_2, \cdots, a_n)$. This amounts to finding the distribution function of $\{\sum_{i=1}^{n}(x_i-a_i)^2\}^{\frac{1}{2}}$. The result involves the integral of $I_s(x)$, multiplied by exponential factors and powers. This integral is reduced to various forms suitable for numerical evaluation. *L. A. Aroian.*

**Birnbaum, Z. W. Effect of linear truncation on a multinormal population.** Ann. Math. Statistics 21, 272–279 (1950).

Define the $(k+l)$-dimensional random variable

$$(X_1, X_2, \cdots, X_k, Y_1, Y_2, \cdots, Y_l)$$

by $(X, Y)$ with a joint probability distribution assumed to be nonsingular multinormal. The $Y_1, Y_2, \cdots, Y_l$ are scores in admission tests administered to all individuals in the $(X, Y)$ population to decide on admission or rejection, and $X_1, X_2, \cdots, X_k$, scores on achievement tests usually administered at a later date. A set of weights $a_j \geqq 0$, $j=1, 2, \cdots, l$, is used to define a composite test score $U=\sum_{j=1}^{l} a_j Y_j$, and a cutting score $\tau$ is chosen so that an individual is admitted if $U \geqq \tau$ and rejected if $U<\tau$. This procedure is said to be linear truncation of $(X, Y)$ in $Y$ to the set $U \geqq \tau$. The author studies the absolute distribution of $X$ if $k=1$. For $k=l=1$, the moments of the absolute distribution of $X$ are found, the determination of $\tau$ for a given expectation or quantile of $X$ after truncation. In case $k=1$, $l$ any value, two problems are discussed: the linear truncation of $(X, Y_1, Y_2, \cdots, Y_l)$ to the set $U \geqq \tau$ for a given expectation or quantile of $X$, minimizing the rejected part of the population; and the linear truncation of $(X, Y_1, Y_2, \cdots, Y_l)$ to the set $U \geqq \tau$ for given expectation of $X$ after truncation, minimizing the variance of $X$ after truncation. *L. A. Aroian.*

**Kimball, Bradford F. On the asymptotic distribution of the sum of powers of unit frequency differences.** Ann. Math. Statistics 21, 263–271 (1950).

"Unit" frequency differences, $u_1, \cdots, u_{n+1}$, are functions of an ordered sample $x_1 < \cdots < x_n$ from a one-dimensional cumulative distribution function $F(x)$; $u_1 = F(x_1)$, $u_{n+1} = 1 - F(x_n)$, $u_{j+1} = F(x_{j+1}) - F(x_j)$, $j=1, \cdots, n-1$. The author considers $y_m = \sum_i u_i^p$, where $p>0$ and the values of $i$ are any $m$ of the integers $1, \cdots, n+1$ $(1 \leqq m \leqq n+1)$. Let

$$z = [(n+1)^p/\sqrt{m}][y_m - E(y_m)],$$

where $E(y_m)$ is the mean of $y_m$. It is shown that as $m, n \to \infty$ so that $(m/n) \to c > 0$, the limiting distribution of $z$ is normal with mean 0 and variance $\Gamma(2p+1) - (1+cp^2)\Gamma^2(p+1)$, except when $p=1$, $c=1$. *D. F. Votaw, Jr.*

**Seth, G. R. On the distribution of the two closest among a set of three observations.** Ann. Math. Statistics 21, 298–301 (1950).

The author obtains the joint distribution of the two closest observations $x'$, $x''$ $(x'<x'')$ of the set $x_1, x_2, x_3$ $(x_1 \leqq x_2 \leqq x_3)$ when the distribution of $x_1, x_2, x_3$ is given or can be obtained. He also obtains the joint distribution of $u=x''-x'$ and $w=(x''-x')/(x_3-x_1)$ in general, and in particular if the density functions of $x_1$ and $x_3$ are given by the same normal probability function. *L. A. Aroian* (Culver City, Calif.).

**Godwin, H. J. On the estimation of dispersion by linear systematic statistics.** Biometrika 36, 92–100 (1949).

Consider the ordered sample $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ from a given population with cumulative distribution function $F(x)$. Let $y_i = x_{i+1} - x_i$. If a linear function of the $x$'s is to be used to estimate the dispersion of the population, it must be a linear combination of the $y$'s. Expressions are obtained for $E(y_i)$ and $E(y_i y_j)$ in terms of $\psi(i) = \int_{-\infty}^{\infty} \{F(x)\}^i \{1-F(x)\}^i dx$ and $\psi(a, b) = \int_{-\infty}^{\infty} \{F(x)\}^a \int_x^{\infty} \{1-F(y)\}^b dy dx$. Making use of these expressions one may obtain the best such estimate (in the sense of minimum variance). The author shows that the range is the best estimate of this type for the rectangular distribution and compares various estimates for the normal distribution for $n=2$ to 10. *H. Chernoff* (Urbana, Ill.).

**Kendall, M. G. Rank and product-moment correlation.** Biometrika 36, 177–193 (1949).

The author investigates the effect of nonnormality in the bivariate universe samples on the expected value and variance of his coefficient of rank correlation, $\tau$, and that of Spearman, $\rho_s$. He supposes the probability distribution function for the universe to be adequately expressed by a limited number of terms of the bivariate Gram-Charlier series which is here written in an operational form analogous to that which leads to Edgeworth's expansion in the univariate case. The calculations necessary to find suitable approximation for these means and variances are carried out. He concludes that the variance of $r_s$, the sample value of $\rho_s$, depends on nonelementary transcendental functions, and discusses the source of the difference between his result and that given by K. Pearson. The Kendall coefficient turns out to be relatively insensitive to departure from normality as compared to the Spearman coefficient and the author discusses and illustrates the use of the former in the estimation of the product moment correlation in nonnormal cases. *C. C. Craig* (Ann Arbor, Mich.).

**Carter, A. H. The estimation and comparison of residual regressions where there are two or more related sets of observations.** Biometrika 36, 26–46 (1949).

Given $p$ treatments, each applied to $n$ locations, with 1 dependent $(Y)$ and $q$ independent $(X_u)$ variables measured at each of the $pn$ plots. Assume the following linear model:

$$Y_{ij} = \mu + \gamma_i + \alpha_j + \sum_{u=1}^{q} \beta_{ui} x_{uij} + \epsilon_{ij},$$

where $\mu$ is the general mean, $\gamma_i$ the added effect of the $i$th treatment, $\alpha_j$ the added effect of the $j$th location, $\beta_{ui}$ the regression effect of $X$ on $Y$ for the $i$th treatment.

$x_{uij}=X_{uij}-\bar{X}_{ui}$, and the $\epsilon_{ij}$ are assumed normally and independently distributed with zero means and common variance $\sigma^2$. The author uses least squares techniques (i) to obtain estimates of the $\beta_{ui}$, (ii) to test hypotheses of the type $\beta_{u1}=\beta_{u2}=\cdots=\beta_{up}=\bar{\beta}_u$, (iii) to test hypotheses of the type $\beta_{ul}=\beta_{um}$ $(l\neq m)$. Detailed computing techniques are given for these special cases: (i) a single independent variate $(q=1)$; (ii) two treatments $(p=2)$; (iii) $q=1$, $p=2$. Examples are given for $q=1$ and $p=2, 4$. The results are contrasted with those given by Yates for $p=2$, in which $\epsilon_{ij}$ has variance $\sigma_i^2$ and $\epsilon_{ij}$ and $\epsilon_{kj}$ are correlated [Proc. Roy. Soc. Edinburgh **59**, 184–194 (1939); these Rev. **1**, 23]. The results in this paper should save some time over the abbreviated Doolittle computing techniques. *R. L. Anderson.*

**Sukhatme, B. V. Random association of points on a lattice.** J. Indian Soc. Agric. Statistics **2**, 60–85 (1949).

Black and white balls are distributed over a rectangular lattice with $mn$ points so that the probability of obtaining a black ball equals $p$ for every lattice point. The author considers the number of triplets of adjacent points under two conditions. The first, termed by the author free sampling, arises when the color of any lattice point is independent of the colors of all other lattice points. The second, termed nonfree sampling, arises when $n_1$ white and $n_2$ black balls $(n_1+n_2=mn)$ are randomly distributed over the lattice. The author considers in each case two different ways of counting the triplets. In the first only triplets occurring on lines parallel to the sides of the lattice are counted. In the second also triplets occurring on lines parallel to the diagonals are included. The author finds the first and second moments for all four resulting distributions. The number of triplets can be used as a test function for testing the randomness of the distribution against the alternative that contagion is present, that is to say that the presence of one black ball increases the probability for finding a black ball at neighboring lattice points. The author finds that the test based on the number of triplets including the diagonals is more efficient than the one excluding the diagonals. The test has been applied to test the randomness of the distribution of diseased plants over a rectangular plantation with $m$ rows and $n$ columns. *H. B. Mann* (Berkeley, Calif.).

**Grundy, P. M. The estimation of error in rectangular lattices.** Biometrics **6**, 25–33 (1950).

The author derives formulae for the intrablock and interblock variances for any number of replications of the rectangular lattice designs introduced by Harshbarger [Virginia Agricultural Experiment Station, Memoir 1 (1947); these Rev. **10**, 202]. *H. B. Mann* (Berkeley, Calif.).

**Federer, W. T. The general theory of prime-power lattice designs.** Biometrics **6**, 34–58 (1950).

The author gives a complete discussion of the analysis of variance of an incomplete $6\times6$ lattice square design. The analysis is illustrated by an example. *H. B. Mann.*

**Nanda, D. N. The standard errors of discriminant function coefficients in plant-breeding experiments.** J. Roy. Statist. Soc. Ser. B. **11**, 283–290 (1949).

This paper derives standard errors of discriminant function coefficients when the variates are themselves subject to errors of measurement as is the case when phenotypic values are used for genotypic variates. Also the standard error for the estimated genetic advance is found. *A. M. Mood.*

**Nanda, D. N. Efficiency of the application of discriminant function in plant-selection.** J. Indian Soc. Agric. Statistics **2**, 8–19 (1949).

Study of the relative efficiency of yield selection versus discriminant function selection using methods given in the paper reviewed above. A numerical example is included. *A. M. Mood* (Santa Monica, Calif.).

**Wold, H. A large-sample test for moving averages.** J. Roy. Statist. Soc. Ser. B. **11**, 297–305 (1949).

Let there be given an autocorrelated time series $\xi_t$ suspected of having been generated from independent variates $\eta_t$ by a process of forming a moving average of span $k$. The author estimates the coefficients for the moving average from the first $k$ autocorrelation coefficients of $\xi$. Then, for any $i$, he adopts a quadratic form in the next $i$ autocorrelations which is transformable to $\chi^2$ as his statistic for testing the hypothesis that $\xi$ was generated from $\eta$ as above. On applying the method to Beveridge's wheat price index the author finds that a quite long time series is needed before such a test can discriminate among alternative hypotheses on the generation of the series, there being so much duplication in the information which neighboring terms of an autocorrelated series provide. *A. Blake.*

**Quenouille, M. H. On a method of trend elimination.** Biometrika **36**, 75–91 (1949).

A method of interlaced parabolas, related to that of E. C. Rhodes [Smoothing, Tracts for Computers, no. 6, Cambridge University Press, 1921]. In discussing how to choose the separation (number of graduated values to be supplied by each parabola) and degree of the parabolas, the author refers to the case when the trend deviations are mutually independent. *H. Wold* (Uppsala).

**Feller, W. Errata.** Ann. Math. Statistics **21**, 301–302 (1950).

Corrections to papers in the same Ann. **19**, 177–189 (1948); **16**, 319–329 (1945); these Rev. **9**, 599; **7**, 459.

**Bjoraa, S. J. Approximate values of premium return assurances.** Skand. Aktuarietidskr. **32**, 176–179 (1949).

In Norway many life annuities are issued with premium return assurances so that when at death the total of the premiums has not been paid out the remaining amount is paid. The exact formula in case of a joint life annuity is rather complicated but an approximate formula,

$$(\bar{a}_{\overline{n}|}-\bar{a}_{\overline{xy}\,\overline{n}|})(1+0.2ri),$$

is suggested, where $r$ is the single premium and $i$ the rate of interest. *P. Johansen* (Copenhagen).

**Delgleize, A. Sur la détermination de $\bar{a}_x$.** Bull. Soc. Roy. Sci. Liège **18**, 391–394 (1949).

The author derives by interpolatory methods two formulae for the value of the continuous whole life annuity. The first terms of his formulae agree with the first terms obtained from the Euler-Maclaurin formula. *E. Lukacs* (Washington, D. C.).

# TOPOLOGY

**Sierpiński, Wacław. Sur un problème de M. Lusin concernant les complémentaires analytiques.** Fund. Math. 36, 44–47 (1949).

A family of sets $\Phi$ is said to have the property $\pi_n$ ($n$ being finite or $\aleph_0$) if, for any sequence of sets $E_i$ ($i=1, 2, \cdots, n$) chosen arbitrarily from $\Phi$, there exists a sequence of sets $H_i$ of $\Phi$ for which we have $\prod_{i=1}^{n} H_i = 0$ and $E_i - \prod_{k=1}^{n} E_k \subset H_i$, $i=1, 2, \cdots, n$. First the author proves the following main theorem: If $\Phi$ is additive and multiplicative (in the finite sense) and has $\pi_2$, then it has also $\pi_n$ for $n=3, 4, \cdots$. This results shows, in particular, that the family $F_1$ of complementary analytic sets [this is an affirmative solution for a problem proposed by N. Lusin [C. R. (Doklady) Acad. Sci. URSS (N.S.) 3 (1934 II), 280–284]] or of sets $PC(A)$ (in a complete separable space), and $F_2$ of Borel sets of additive class $\alpha$ ($\alpha>0$) (in a metric space) have $\pi_n$ for $n=2, 3, 4, \cdots$. Then the author gives an example which shows that we cannot extend the above theorem to the case $n=\aleph_0$. The problem of whether the family $F_1$ or $F_2$ has $\pi_{\aleph_0}$ is left open. Finally the author shows by an example that there exist three analytic sets $E_1, E_2, E_3$ for which no Borel sets $Q_1, Q_2, Q_3$ have the property: $E_i - E_1 \cdot E_2 \cdot E_3 \subset Q_i$ ($i=1, 2, 3$) and $Q_1 \cdot Q_2 \cdot Q_3 = 0$, which answers negatively another problem of N. Lusin [loc. cit.]. *K. Kunugi* (Osaka).

**Figueras Calsina, E. Concerning the non-equivalence of Menger's and Urysohn's definitions of dimension.** Revista Mat. Hisp.-Amer. (4) 9, 53–58 (1949). (Spanish)

The author considers three definitions for the dimension of a topological space: an intrinsic definition due to P. Urysohn [C. R. Acad. Sci. Paris 175, 440–442 (1922); Fund. Math. 7, 30–137 (1925)], the usual intrinsic definition due to K. Menger [W. Hurewicz and H. Wallman, Dimension Theory, Princeton University Press, 1941, p. 24; these Rev. 3, 312], and Menger's original "extrinsic" definition [Monatsh. Math. Phys. 33, 148–160 (1923)]. It is known that these three definitions are equivalent for separable metric spaces; however, they need not be equivalent for more general spaces. There is an example due to L. W. Cohen [C. R. Acad. Sci. Paris 184, 1368–1370 (1927)] for which the Urysohn dimension and Menger's intrinsic dimension are the same, but the extrinsic dimension of Menger is different from these two. In this note the author gives two examples of spaces for which Menger's intrinsic dimension and Urysohn's dimension are different. *W. S. Massey.*

**Smirnov, Yu. M. On topological spaces compact in a given interval of powers.** Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 155–178 (1950). (Russian)

Let $a$ and $b$ be any infinite cardinal numbers, and let $a$ be not greater than $b$. A topological space $R$ is said to be $[a, b]$-compact if (1) for every regular cardinal number $m$ such that $a \leqq m \leqq b$, every subset $M$ of $R$ of cardinal number $m$ admits a complete limit point $\xi$ (i.e., a point $\xi$ such that every neighborhood of $\xi$ intersects $M$ in a set of cardinal number equal to the cardinal number of $M$). This notion was introduced by Alexandroff and Urysohn [see Verh. Akad. Wetensch. (Amsterdam) Afd. Natuurk. Sect. 1. 14, no. 1 (1929)]. In the present paper this definition and its ramifications are thoroughly explored. It is proved first that $[a, b]$-compactness is equivalent to each of the following properties: (2) in $R$, every family of nonvoid closed sets $F_1 \supset F_2 \supset \cdots \supset F_\alpha \supset \cdots \supset F_\alpha \supset \cdots$ ($\alpha<\theta$) well-ordered by set-inclusion and having a regular ordinal number $\theta$ such that $\omega(a) \leqq \theta \leqq \omega(b)$ has nonvoid intersection; (3) every family of closed subsets of $R$ satisfying the conditions of (2), and with the additional property that every subfamily of cardinal number $<a$ has nonvoid intersection, has total intersection nonvoid; (4) every open covering of $R$ having a regular cardinal number $m$ such that $a \leqq m \leqq b$ admits a subcovering of cardinal number less than $a$. No fewer than eight variations on these four properties are defined and discussed. Locally $[a, b]$-compact spaces are defined in the usual way; it is shown that an arbitrary locally $[a, b]$-compact Hausdorff space $R$ can be imbedded in a $[a, b]$-compact Hausdorff space by the adjunction of a single point $\xi$; and that among all such extensions of $R$, the usual extension, where neighborhoods of $\xi$ are complements of closed $[a, b]$-compact subsets of $R$, possesses the strongest possible topology. A topological space $R$ is said to be hereditarily $[a, b]$-compact if all of its subsets are $[a, b]$-compact. The equivalence of the following properties of a topological space $R$ is proved: (5) $R$ is hereditarily $[a, b]$-compact, the cardinal interval $[a, b]$ containing at least one regular cardinal number; (6) $R$ is hereditarily $[a, c]$-compact for all $c \geqq a$; (3) every family of closed subsets of $R$, $F_1 \supset F_2 \supset \cdots \supset F_\alpha \supset \cdots \supset F_\alpha \supset \cdots$ ($\alpha<\theta$), well-ordered by set-inclusion and with regular ordinal number $\theta$, contains a set $F_{\alpha_0}$ such that $F_\alpha = F_\beta$ for all $\alpha, \beta > \alpha_0$. Some simple analogues of the Cantor-Bendixson theorem are proved. The paper closes with the result that every topological space which is hereditarily $[\aleph_{\alpha+1}, b]$-compact for some $b \geqq \aleph_{\alpha+1}$ has cardinal number not greater than $2^{\aleph_\alpha}$. *E. Hewitt.*

**Ivanov, A. A. Isotopy of compacta in Euclidean spaces.** Doklady Akad. Nauk SSSR (N.S.) 71, 1021–1022 (1950). (Russian)

This note announces and sketches a proof of the following assertion, and constitutes a kind of extension of the celebrated example of Antoine. If $M$ is a compact set in $n$-dimensional Euclidean space $E^n$ ($n>3$) and lies in an $(n-1)$-dimensional Euclidean subspace $E^{n-1}$, then, in order that there exist a compact set in $E^n$ which is homeomorphic but not isotopic to $M$ in $E^n$, it is necessary and sufficient that $M$ be uncountable.

*L. Zippin* (Flushing, N. Y.).

**Henkin, Leon. A problem on inverse mapping systems.** Proc. Amer. Math. Soc. 1, 224–225 (1950).

Soit $A$ un ensemble ordonné filtrant; on sait que la notion de limite projective d'une famille $(M_\alpha)$, $\alpha \in A$, d'ensembles est définie par la donnée d'une famille d'applications $(f_{\alpha\beta})$, où $f_{\alpha\beta}$ est une application de $M_\beta$ sur $M_\alpha$ (pour $\alpha<\beta$) telle que $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ pour $\alpha<\beta<\gamma$; la limite projective de la famille $(M_\alpha)$ est la partie $M$ du produit $\prod_{\alpha \in A} M_\alpha$ formée des $(x_\alpha)$, $\alpha \in A$, tels que $f_{\alpha\beta}(x_\beta) = x_\alpha$ pour $\alpha<\beta$. Moyennant certaines hypothèses sur les $M_\alpha$ et les $f_{\alpha\beta}$, on connaît des cas importants où $M$ n'est pas vide. Mais l'auteur montre que pour un ensemble filtrant $A$ donné, on peut toujours construire explicitement une famille $(M_\alpha)$ d'ensembles et une famille $(f_{\alpha\beta})$ d'applications telle que la limite projective correspondante soit vide, sauf dans les deux cas suivants: (1) $A$ possède un plus grand élément; (2) $A$ contient une suite dénombrable cofinale à $A$.

*J. Dieudonné* (Nancy).

**Floyd, E. E. Some characterizations of interior maps.** Ann. of Math. (2) **51**, 571–575 (1950).

Let $f: X \to Y$ be a map onto where $X$, $Y$ are Peano continua and $\dim f^{-1}(y) = 0$ for each $y$ in $Y$. The author shows (among other things) that each of the following conditions is both necessary and sufficient that $f$ be interior [=open]. (A) For each open connected set $V$ in $Y$ and each component $U$ of $f^{-1}(V)$ the induced homomorphism $f^*: H^1(\bar{V}, \bar{V} - V) \to H^1(\bar{U}, \bar{U} - U)$ is an isomorphism into. (B) For each map $h: (0, 1) \to Y$ and each $x$ in $f^{-1}h(0)$ there exists a map $g: (0, 1) \to X$ such that $fg = h$ and $g(0) = x$. In (A) $H^1$ denotes the integral Čech cohomology group. Both (A) and (B) are related to results of Whyburn [Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942; these Rev. 4, 86]. In connection with (B) see also Stoïlow [Leçons sur les principes topologiques de la théorie des fonctions analytiques, Gauthier-Villars, Paris, 1938]. Under (A) the author's result is actually more general. It is enough to assume that $f$ is quasi-monotone and the condition $\dim f^{-1}(y) = 0$ may be dropped.

*A. D. Wallace* (New Orleans, La.).

**Schwartz, Marie-Hélène. Applications intérieures régulières dans les variétés à $n$ dimensions.** C. R. Acad. Sci. Paris **230**, 1244–1245 (1950).

Let $U$ and $E$ be connected and orientable manifolds of $n$ dimensions, $E$ compact and $U$ noncompact. A regular interior mapping of $U$ into $E$ is a locally one-to-one and bicontinuous mapping, with the exception of a complex of $n-2$ dimensions. The exceptional complex is subdivided and local topological degrees are defined for each subcomplex. The author first establishes a formula between such degrees and certain Euler characteristics and, then, by imposing a Riemann metric on the manifold, derives a formula from the generalised Gauss-Bonnet formula which contains the first formula. *S. Chern* (Chicago, Ill.).

**Schwartz, Marie-Hélène. Applications $A$-intérieures et formule de Gauss-Bonnet généralisée.** C. R. Acad. Sci. Paris **230**, 1337–1338 (1950).

This is a continuation of the note reviewed above. An interior mapping is called $A$-interior if it satisfies a condition expressing the way that the noncompact manifold is exhausted by a family of compact manifolds. The Gauss-Bonnet formula is applied to each compact manifold of the family and, under certain assumptions, an asymptotic formula is established. *S. Chern* (Chicago, Ill.).

**Schwartz, Marie-Hélène. Application $A$-intérieure et théorie des défauts.** C. R. Acad. Sci. Paris **230**, 1376–1378 (1950).

The results of the two notes reviewed above are applied to study relations between local defects, thus generalizing the work of Nevanlinna and Ahlfors to higher dimensions. Several asymptotic formulas are stated. *S. Chern.*

**Sitnikov, K. On continuous mappings of a Euclidean space.** Doklady Akad. Nauk SSSR (N.S.) **71**, 621–623 (1950). (Russian)

The author sketches a proof of the following natural extension of a theorem of Borsuk [Fund. Math. **21**, 236–243 (1933)]. Let $\varphi$ be a continuous map of the $n$-dimensional Euclidean space $R^n$ into itself in which the counter-image $\varphi^{-1}y \subset R^n$ of each point $y \varepsilon \varphi R^n$ has a diameter less than a fixed $\alpha > 0$. Then all the Betti groups of the open set $\varphi R^n$ are null-groups. The first step in this proof is the result that the map $\varphi$ is of covering-order $\gamma$ at every point of $\varphi R^n$, where the constant $\gamma = \pm 1$. To achieve this, the author uses a simplicial approximation $f$ to $\varphi$, defined on a large simplex $T^n$ suitably subdivided; and $T^n$ and $fT^n$ are regarded as polyhedra in $R^n$. Then he uses an imbedding of $R^n$ into $R^{2n+1}$ in order to exploit a theorem of Kuratowski [Ann. Soc. Polon. Math. **17**, 118–119 (1938)] in which there appears a topological image $H$ of the polyhedron $fT^n$ such that $g = Hf$ is a map of $T^n$ that "moves" no point farther than the given $\alpha > 0$.

From the result that $\gamma = \pm 1$ it follows easily that $\varphi R^n$ is open and that compacta in $\varphi R^n$ have compact counter-images. The remainder of the proof, involving cycles and singular cycles, uses some earlier work of the author [same Doklady (N.S.) **66**, 1059–1062 (1949); these Rev. **11**, 45].

*L. Zippin* (Flushing, N. Y.).

**Aleksandrov, P., and Sitnikov, K. On continuous mappings of closed manifolds.** Doklady Akad. Nauk SSSR (N.S.) **71**, 821–823 (1950). (Russian)

This note considers $\alpha$-maps (inverse-sets of diameter less than a given $\alpha > 0$) in the light of a theorem of Kuratowski cited in the preceding review. The following theorems are stated, the first as easy, the second with some discussion. (1) Let $X$ be a compact neighborhood retract lying in a Euclidean or a Hilbert space. There is an $\alpha > 0$, such that every $\alpha$-map of $X$ into any compactum $Y$ induces an isomorphic map of the Betti groups of $X$ into the corresponding groups of $Y$. Here one can choose for $\alpha$ any number such that an $\alpha$-neighborhood $O(X, \alpha)$ of the set $X$ in $R$ can be retracted to $X$. (2) If $X$ is an oriented $n$-dimensional closed pseudo-manifold lying in a Euclidean or a Hilbert space, then there is an $\alpha$ such that from the existence of an $\alpha$-map $f$ of $X$ onto any closed pseudomanifold $Y$ it follows that $Y$ is orientable and that $f$ is of degree $\pm 1$. Here $\alpha$ can be chosen as any positive number less than the "existence-measure" of the basic $n$-dimensional cycle of the pseudo-manifold $X$. If $X$ is an $n$-dimensional homological orientable manifold, then there is an $\alpha > 0$ such that an $\alpha$-map $f$ of $X$ onto an arbitrary homology manifold $Y$ implies that $X$ and $Y$ are homologically equivalent and that $f$ induces an isomorphism of the Betti groups of $X$ and corresponding groups of $Y$. Here $\alpha$ can be chosen as any positive number such that an $\alpha$-neighborhood of $X$ can be retracted to $X$ [obviously every such $\alpha$ is less than the existence-measure of the basic cycle of the manifold $X$]. *L. Zippin.*

**Postnikov, M. M. Classification of continuous mappings of an $(n+1)$-dimensional complex into a connected topological space which is aspherical in dimensions less than $n$.** Doklady Akad. Nauk SSSR (N.S.) **71**, 1027–1028 (1950). (Russian)

This note considers the homotopy classification problem for maps of an $(n+1)$-dimensional complex $K$ into a pathwise connected topological space $Y$ which is $(n-1)$-connected, $n > 2$; i.e., the homotopy groups $\pi_p(Y)$ are trivial if $p < n$. The case $n = 2$ has been considered previously by the author [same Doklady (N.S.) **64**, 461–462 (1949); these Rev. **10**, 560] and Whitney [Ann. of Math. (2) **50**, 270–284 (1949); these Rev. **11**, 531]. Let $\eta: \pi_n(Y) \to \pi_{n+1}(Y)$ denote the homomorphism defined by composing a map $S^n \to Y$ which represents an element $\alpha \varepsilon \pi_n(Y)$ with an essential map $S^{n+1} \to S^n$ to obtain a map $S^{n+1} \to Y$ which represents the element $\eta(\alpha) \varepsilon \pi_{n+1}(Y)$. Define a "multiplication" which associates with any two elements $\alpha, \beta \varepsilon \pi_n(Y)$ an element $\alpha\beta \varepsilon \pi_{n+1}(Y)$ so

as to satisfy the following four conditions: $(\alpha+\beta)\gamma=\alpha\gamma+\beta\gamma$, $\alpha\beta=\beta\alpha$, $2\alpha\beta=0$, and $\alpha\alpha=\eta(\alpha)$. It may be shown that such a multiplication always exists. If one now applies the definitions of Steenrod [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154], there are defined homomorphisms $Sq_i: H^p(K, \pi_n(Y))\to H^{p+i}(K, \pi_{n+1}(Y))$. It may be shown that the homomorphisms thus defined are independent of the choice of the multiplication used.

Let $f, g: K\to Y$ be two maps which are $n$-homotopic, i.e., the partial maps $f|K^n$ and $g|K^n$ are homotopic (here $K^n$ denotes the $n$-skeleton of $K$). Let $f', g': K\to Y$ be maps such that $f'|K^n=g'|K^n$, $f'$ is homotopic to $f$, and $g'$ is homotopic to $g$. Then the separation cocycle $d^{n+1}(f', g')$ is defined [see Eilenberg, Ann. of Math. (2) **41**, 231–251 (1940), p. 237; these Rev. **1**, 222]. Let $\mathbf{d}^{n+1}(f', g')$ denote its cohomology class. The set of elements $\mathbf{d}^{n+1}(f', g')$ for all possible choices of $f'$ and $g'$ satisfying the conditions stated above may be denoted by $\mathbf{O}^{n+1}(f, g)$; it is a subset of the cohomology group $H^{n+1}(K, \pi_{n+1})$. It is known that the maps $f$ and $g$ are homotopic if and only if the set $\mathbf{O}^{n+1}(f, g)$ contains the zero element of $H^{n+1}(K, \pi_{n+1})$. The author's main result is that $\mathbf{O}^{n+1}(f, g)$ is a coset of the subgroup $Sq_{n-3}[H^{n-1}(K, \pi_n)]$. Since conditions are already known which enable one to decide whether or not two given maps $f, g: K\to Y$ are $n$-homotopic, this result enables one to give an enumeration of the homotopy classes of maps $K\to Y$. The author also states an analogous theorem about obstructions to extensions of continuous maps, and asserts that his main result is derived from this theorem on extensions. No proofs are given. *R. H. Fox and W. S. Massey.*

**Pontryagin, L. S. Classification of the mappings of an $(n+1)$-dimensional sphere into a polyhedron $K_n$ whose fundamental group and Betti groups of dimensions $2, \cdots, n-1$ are trivial.** Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 7–44 (1950). (Russian)

This paper is an exposition, including complete proofs, of results previously announced by the author [Doklady Akad. Nauk SSSR (N.S.) **65**, 797–800 (1949); these Rev. **11**, 122]. *R. H. Fox and W. S. Massey.*

**Begle, Edward G. The Vietoris mapping theorem for bicompact spaces.** Ann. of Math. (2) **51**, 534–543 (1950).

The classical Vietoris homology theory of compact metric spaces is generalized to the bicompact Hausdorff spaces by considering their (finite open) coverings instead of the metric. By means of this extended theory, the author proves in this paper a generalization of the Vietoris mapping theorem in the following form. Let $X$ and $Y$ be two bicompact Hausdorff spaces. For each covering $M$ of $X$ and each subset $W$ of $X$, let $X(M)\cap W$ denote the simplicial complex consisting of all simplexes $\sigma$ such that all vertices of $\sigma$ are contained in $U\cap W$ for some open set $U$ of $M$. A mapping $f$ of $X$ onto $Y$ is a Vietoris mapping of order $n$ if for each covering $M$ of $X$ and each point $y$ of $Y$ there is a covering $P$ of $X$ which is a refinement of $M$ such that any $k$-cycle, $0\leqq k\leqq n$, on $X(P)\cap f^{-1}(y)$ bounds on $X(M)\cap f^{-1}(y)$. Then the generalized Vietoris mapping theorem states that the homomorphism of the Vietoris homology group $H_*^n(X)$ into $H_*^n(Y)$ induced by a Vietoris mapping $f: X\to Y$ of order $n$ is an isomorphism onto. If the coefficient group is either an elementary compact topological group or a field, then the theorem can be put in a more convenient form as follows. If $f$ is a mapping of $X$ onto $Y$ such that for each point $y$ of $Y$, and for each integer $k$, $0\leqq k\leqq n$, $H_*^k(f^{-1}(y))=0$, then the induced homomorphism $f^*: H_*^n(X)\to H_*^n(Y)$ is an isomorphism onto. *S. T. Hu* (Princeton, N. J.).

**Brändli, E. R., und Eckmann, B. Cartesisches und Alexandersches Produkt in der Cohomologietheorie.** Comment. Math. Helv. **24**, 68–72 (1950).

A somewhat simplified proof of the result in part III of Brändli's thesis [Beiträge zur Theorie des Cohomologieringes, Eidgenössische Technische Hochschule, Zürich, 1948; these Rev. **11**, 47] on a connection between Cartesian products and cup products. *J. Dugundji.*

**Hu, Sze-Tsen. On a general homotopy problem of Eilenberg.** Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 267–272 (1949).

Let $Y$ be a topological space, $K$ a finite Euclidean complex, and $K_0$ a closed sub-complex. Denote the $n$-skeleton of $K$ by $K^n$, and let $\bar{K}^n=K^n\cup K_0$. Two maps $f, g: K\to Y$ are said to be $n$-homotopic relative to $K_0$ if the maps $f|\bar{K}^n$ and $g|\bar{K}^n$ are homotopic relative to $K_0$. The author restricts his attention to the case where $Y$ is pathwise-connected and simple in the dimension $n$, i.e., the fundamental group $\pi_1(Y)$ operates trivially on the homotopy group $\pi_n(Y)$. Suppose that $f, g: K\to Y$ are $(n-1)$-homotopic relative to $K_0$. Let $h_t: \bar{K}^{n-1}\to Y$, $0\leqq t\leqq 1$, be a homotopy such that $h_0=f|\bar{K}^{n-1}$, $h_1=g|\bar{K}^{n-1}$, and $h_t(x)=f(x)=g(x)$ for $x\in K_0$ and $0\leqq t\leqq 1$. The author shows how to associate with each such homotopy an element $\delta^n(h_t)$ of the relative cohomology group $H^n(K, K_0)$, using the homotopy group $\pi_n(Y)$ as coefficient group; $\delta^n(h_t)$ is the "obstruction" to extending the homotopy $h_t$ one dimension. Let $\mathbf{O}^n(f, g)$ denote the set of all such elements $\delta^n(h_t)$ for all homotopies $h_t$ of the type described. Then the main results are the following. (a) The maps $f$ and $g$ are $n$-homotopic if and only if $\mathbf{O}^n(f, g)$ contains the zero element of $H^n(K, K_0)$. (b) $\mathbf{O}^n(f, f)=\mathbf{O}^n(g, g)$ is a subgroup of $H^n(K, K_0)$. (c) $\mathbf{O}^n(f, g)$ is a coset of $\mathbf{O}^n(f, f)$. *W. S. Massey* (Providence, R. I.).

* **Whitehead, J. H. C. On simply connected, 4-dimensional polyhedra.** Topologie algébrique, pp. 103–106. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

An apparently belated appearance of a note announcing the main results of a more extensive paper already reviewed [Comment. Math. Helv. **22**, 48–92 (1949); these Rev. **10**, 559]. *J. Dugundji* (Los Angeles, Calif.).

* **Leray, Jean. L'homologie filtrée.** Topologie algébrique, pp. 61–82. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

This is a resumé of a course of lectures given by the author in 1947–48 with various refinements suggested by H. Cartan. The word homology is used throughout to replace cohomology. Let $X$ be a locally compact space to which there is associated a system $\mathfrak{B}$ of differential rings $\mathfrak{B}(F)$, $F$ being an arbitrary closed subset of $X$. The $\mathfrak{B}(F)$ are connected by certain homomorphisms and satisfy certain conditions of continuity. There is determined by $X$, $\mathfrak{B}$ a ring denoted by $\mathfrak{H}(X\bigcirc\mathfrak{B})$ called the homology ring of $X$ with respect to $\mathfrak{B}$. Something like dimension or degree is introduced by means of filtrations: a ring $\mathfrak{A}$ is filtered if there exists a mapping $f$ (filtration) of $\mathfrak{A}$ into the real numbers augmented by $+\infty$ such that $f(a-a')\geqq\min[f(a), f(a')]$, $f(aa')\geqq f(a)+f(a')$, $f(0)=+\infty$. Suppose now that the rings $\mathfrak{B}(F)$ are filtered.

Let $l$ be a positive or negative integer. There is defined a homology ring $\mathfrak{C}(X^l \circ \mathfrak{G})$ with filtration depending on $l$. In addition there is defined a sequence of filtered differential rings $\mathfrak{C}_r(X^l \circ \mathfrak{G})$, $r=l+1, l+2, \cdots$. Also, $\mathfrak{C}_r$ is graduated, that is, each element of $\mathfrak{C}_r$ is the sum of homogeneous elements of definite degrees; $\mathfrak{C}_r$ admits a differential $\delta_r$ which increases the degrees of homogeneous elements by $r$. The homology ring of the differential ring $\mathfrak{C}_r$ is $\mathfrak{C}_{r+1}$. If the rings $\mathfrak{G}(F)$ of the system $\mathfrak{G}$ are independent of $F$, then $\mathfrak{C}(X^l \circ \mathfrak{G})$ and $\mathfrak{C}_r(X^l \circ \mathfrak{G})$ are topological invariants. When $l=1$, and when the filtration and differential of $\mathfrak{G}$ are nul, the rings $\mathfrak{C}_r$ reduce to $\mathfrak{C}(X \circ \mathfrak{G})$ filtered by the dimension function. Let $\phi$ be a continuous mapping of $X$ onto a locally compact space $Y$ and let $l, m$ be integers with $l<m$. Associated with $\phi$, $\mathfrak{G}$, $l$, $m$ are topological invariants denoted by $\mathfrak{C}(\phi^{-1}Y^m \circ X^l \circ \mathfrak{G})$ and $\mathfrak{C}_r(\ )$, $r=m+1, \cdots$. The filtered ring $\mathfrak{C}_{m+1}$ is isomorphic to $\mathfrak{C}(Y^m \circ \phi\mathfrak{F}_m)$ where the system $\phi\mathfrak{F}_m$ is the image by $\phi$ of a certain system $\mathfrak{F}_m$ in $X$ depending on $\mathfrak{G}$, $l$, $m$. *P. A. Smith* (New York, N. Y.).

***Cartan, Henri, et Leray, Jean. Relations entre anneaux d'homologie et groupes de Poincaré.** Topologie algébrique, pp. 83–85. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

Let $\Gamma$ be a finite group of continuous automorphisms of a locally compact space $X$ without fixed points. By considering equivariant phenomena in terms of the homology theory of the preceding review, the authors show briefly how to obtain relations between the invariants of $X$ and those of the space of orbits of $\Gamma$. *P. A. Smith.*

***Ehresmann, Charles. Sur la théorie des espaces fibrés.** Topologie algébrique, pp. 3–15. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

Definitions are given of fiber bundles and certain relations between them; in particular, of one being "subordonnée" to another. Theorems about these notions are applied in the rest of the paper. Using $n$ complex coordinates and corresponding transformations, a definition of "almost complex" structure of a differentiable manifold $V_{2n}$ is given; this must exist if $V_{2n}$ is to have a complex analytic structure. An almost complex structure exists if and only if there is a differential form in $V_{2n}$ of degree 2 and everywhere of rank $2n$. This holds for $2n=6$ if the third homology group of $V_{2n}$ vanishes (for instance, for the 6-sphere). It fails for the 4-sphere. *H. Whitney* (Cambridge, Mass.).

***Hodge, W. V. D. The finite algebraic form of the theory of harmonic integrals.** Topologie algébrique, pp. 43–54. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

If $K$ is a covering complex of a manifold $M$ a real-valued function $\Phi(E_i^p)$ is defined by the values of a $p$-fold integral $P$ over the cells $E_i^p$. Thus $P$ is a $(p, R)$ chain of $K$, $R$ being the field of real numbers. The conditions for an integral to be harmonic are reinterpreted as conditions on $p$-chains as follows: $\Phi(p, G)$ is harmonic if (1) $\Phi$ is a cocycle, and (2) an "associated" chain $\Phi^*(p, G^*)$, where $G^*$ is the character group of $G$, is a cycle. (The relationship between $\Phi$ and $\Phi^*$ is subject to certain conditions which can be satisfied, for instance, if $G$ (and $G^*$) is the additive group of real numbers.) The harmonic $(p, G)$-chains $\Phi$ form an additive group isomorphic with a subgroup of the cohomology group $Z_p(K, G)$ and the associated $(p, G^*)$-chains $\Phi^*$ are similarly related to the homology group $Z^p(K, G^*)$. Examples and problems are discussed, and it is stated that by suitable choice of $G$ it is possible to find deeper results in the theory of manifolds than those obtained for $G=R$ in the author's "The Theory and Applications of Harmonic Integrals" [Cambridge University Press; Macmillan, New York, 1941; these Rev. 2, 296]. *D. B. Scott* (London).

## GEOMETRY

***Lebesgue, Henri. Leçons sur les Constructions Géométriques.** Gauthier-Villars, Paris, 1950. vi+304 pp.

This is the last course of lectures given by Lebesgue before his death in 1941. The ramifications of the topic are eminently suited to introduce the student to various branches of modern mathematics. This has been achieved in a manner reminiscent of F. Klein's courses of lectures; the book is written in the author's well-known lucid style. The wealth of material treated may be seen from the following brief summary of the contents. Part I. Trisection of the angle; duplication of the cube. Constructions with ruler and compass; with compass only; with ruler only; with ruler and set square; with ruler and transferer of distances (Eichmass); with parallel ruler; and with linkages in general. Part II. Algebraic theory of constructions, especially of those with ruler only, with ruler and compass, and with ruler and transferer of distances; the regular polygons and the cyclotomic equation. Continued fractions and the irrationality of $e$ and $\pi$. Transcendence of $e$ and $\pi$. Part III. Pointwise construction of curves with the ruler only; rational points on algebraic curves; Lüroth's theorem on unicursal curves; Riemann surface and genus; Nöther's theorem. Applications to conics and cubic curves. *F. A. Behrend.*

**Hjelmslev, Johannes. On the foundations of the geometry of the compass.** Mat. Tidsskr. B. 1949, 65–84 (1949). (Danish)

This paper treats the question of whether the results on construction with the compass alone can be obtained without Eudoxos' axiom. For this purpose a system of axioms using point, pair of points and congruence of pairs of points as basic concepts is given. The circle with center $P$ and radius $AB$ consists of the points $X$ for which $PX$ is congruent to $AB$. There are axioms which guarantee that the points on a circle obey the laws of cyclic order, the existence of intersections of circles, the distinction between interior and exterior points of a circle. A motion is a mapping of the plane on itself which carries each pair of points into a congruent pair. The existence of reflections, that is of exactly one motion which is not the identity and leaves two given points fixed, is postulated. Straight lines are defined and their properties developed. Euclidean geometry is obtained by postulating that laying off the radius six times on a circle leads back to the starting point. It is shown that these axioms suffice to define and derive the main properties of inversions. If the requirement is added that two lines without a common normal intersect, then Mohr's result can be

established, that all constructions with ruler and compass can be accomplished by the use of the compass alone. Since the last axiom does not properly belong to the geometry of circles, it is shown that it suffices to require instead that there always exists a circle intersecting three given circles.

*H. Busemann* (Los Angeles, Calif.).

**Zacharias, Max. Bemerkungen zu der Arbeit von O. Nehring "Über ein Dreiecksproblem."** J. Reine Angew. Math. 187, 129–130 (1950).

Zacharias points out that Nehring's paper [J. Reine Angew. Math. 186, 70–77 (1944); these Rev. 7, 165] deals mainly with known properties of the geometry of the triangle which Nehring fails to identify as such, and Nehring uses lengthy analytical methods to prove propositions which can be proved synthetically in a few lines. The last of Nehring's propositions, which he left unproved on account of his proof being "troublesome," is generalized by Zacharias as follows. Given two points $P, Q$ in the plane of a triangle $A_i$, the pedal triangles of $P, Q$ for $A_i$ are orthological if and only if the points $P, Q$ are collinear with the circumcenter of the triangle $A_i$. A simple synthetic proof is provided.

*N. A. Court* (Norman, Okla.).

**Varoli, Giuseppe. Una proposizione di G. A. Kinner ed il problema della trisezione dell'angolo.** Boll. Un. Mat. Ital. (3) 5, 78–81 (1950).

**Bonferroni, Carlo. Un teorema sul triangolo e il teorema di Napoleone.** Boll. Un. Mat. Ital. (3) 5, 85–89 (1950).

**Gougenheim, André. Sur une propriété des segments capables sphériques décrits sur un quart de grand cercle.** C. R. Acad. Sci. Paris 230, 1385–1386 (1950).

A "segment capable" on a spherical surface is defined as the locus of points such that the great circles joining them to two fixed points meet at a fixed angle. It is shown that, if the two fixed points are extremities of a quarter of a great circle, the locus is a curve which divides the entire spherical surface into two equal and superposable parts. It is suggested that the use of this method of subdivision in world map projections would offer less distortion than the conventional hemispheres.

*T. N. E. Greville.*

**Császár, Ákos. A polyhedron without diagonals.** Acta Univ. Szeged. Sect. Sci. Math. 13, 140–142 (1949).

The author constructs, in 3-dimensional space, a polyhedron with plane triangular faces and straight edges such that each pair of vertices is joined by an edge. This polyhedron has seven vertices and is homeomorphic to the torus. He suggests the problem of showing how to construct such a polyhedron (not necessarily homeomorphic to the torus) having any given number ($\geqq 4$) of vertices. The case of four vertices is of course trivial.

*W. T. Tutte.*

**Hohenberg, Fritz. Die linearen und quadratischen Gebilde der komplexen affinen Ebene.** Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 157, 177–236 (1949).

Die komplexe affine Ebene $E$ wird auf eine reelle vierdimensionale Mannigfaltigkeit abgebildet und zwar in verschiedener Weise. Es ergibt sich so ein anschaulicher Überblick über die komplexen Geraden und Kegelschnitte, über die Segreschen Hyperkegelschnitte und über die linearen und quadratischen Kettengebilde in $E$. Ist $P=(z_1, z_2)$ ein komplexer Punkt in $E$, $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, so wird bei einer der Abbildungen $P$ dem Punkt $(x_1, y_1, x_2, y_2)$ im reellen affinen vierdimensionalen Raum zugeordnet. Eine andere Abbildung geht folgendermassen hervor: $E_1$ sei die Gauszsche Zahlenebene $z_1$, $E_2$ die Ebene $z_2$; man wählt im dreidimensionalen Raume $(x, y, z)$ für $E_1$ die Ebene $z=1$, für $E_2$ die Ebene $z=-1$ und ordnet $P$ die Verbindungsgerade der Bildpunkte zu. Der Verf. gibt eine ausführliche und sorgfältige Betrachtung der genannten Figuren bei diesen Abbildungen und bemerkt dass die quadratischen Kettengebilden sich noch nicht in der Literatur vorzufinden scheinen.

*O. Bottema* (Delft).

**Ladopoulos, P. D. Contribution to projective geometry. The angle between two conics.** Bull. Soc. Math. Grèce 24, 76–84 (1949). (Greek. French summary)

In the projective plane denote by $\varkappa(a, b, c, d)$ the cross ratio of the 4 lines through $x$ and the points $a, b, c, d$, respectively. If a point $x$ varies on a conic $C_i$ in the quadrangular pencil with base points $a, b, c, d$, then $\varkappa(a, b, c, d)$ has a constant value $\alpha_i$. The cross ratio of four conics $C_1, C_2, C_3, C_4$ in the pencil is defined by

$$[C_1, C_2; C_3, C_4]=\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_3}\Big/\frac{\alpha_4-\alpha_1}{\alpha_3-\alpha_4}.$$

If $C_1$ and $C_2$ are any two conics intersecting at four points and $C', C''$ are the conics through the circular points in the pencil determined by $C_1$ and $C_2$, then, in analogy to Laguerre's definition of angle, $(2\pi i)^{-1}\log[C_1, C_2; C', C'']$ is defined as the angle between $C_1$ and $C_2$. The angle is 0 if the four intersections of $C_1$ and $C_2$ lie on a circle. Distinguish three points 1, 2, 3 and delete the lines 12, 23, 31 from the plane. The conics through 1, 2, 3 are called pseudolines and are shown to satisfy the axioms for the straight lines in Euclidean geometry; for instance, parallel (=nonintersecting) pseudolines intersect a pseudoline not parallel to them at the same angle. The sum of the angles in a triangle is zero, etc.

*H. Busemann* (Los Angeles, Calif.).

**Seiden, Esther. A theorem in finite projective geometry and an application to statistics.** Proc. Amer. Math. Soc. 1, 282–286 (1950).

Let $m_t(r, s)$ be the maximum number of points of the finite analytic projective geometry $PG(r-1, s)$ of $GF(s)$ of $r-1$ dimensions such that no $t$ of the points lie in a subspace of $t-2$ dimensions. R. C. Bose [Sankya 8, 107–166 (1947); these Rev. 10, 201] proved that $m_3(4, s)=s^2+1$ if $s$ is odd and $s^2+1\leqq m_3(4, s)\leqq s^2+s+2$ if $s$ is even. The main result of the author is $m_3(4, 4)\geqq 17$. This together with Bose's result gives $m_3(4, 4)=17$.

*H. B. Mann* (Berkeley, Calif.).

**Snapper, Ernst. Periodic linear transformations of affine and projective geometries.** Canadian J. Math. 2, 149–151 (1950).

Given a field $K$, each of the following is associated with a vector space of dimension $s$ over $K$: (1) an affine geometry $A_s$ of dimension $s$, (2) a projective geometry $P_{s-1}$ of dimension $s-1$, (3) an algebra $R_s$ of degree $s$ over $K$. A multiplication by a nonsingular element of $R_s$ may be associated via the vector space with a collineation in $A_s$ or $P_{s-1}$. In particular, roots of unity in $R_s$ yield periodic collineations in $A_s$ and $P_{s-1}$. Thus if $K$ is a finite field and $A_s$ is the finite field of degree $s$ over $K$, the existence of a primitive root in $A_s$ may be used to prove a theorem of Singer [Trans. Amer. Math. Soc. 43, 377–385 (1938)] and a later theorem of Bose [J. Indian Math. Soc. (N.S.) 6, 1–15 (1942); these Rev. 4, 33].

*M. Hall, Jr.* (Columbus, Ohio).

**Schilling, Friedrich. Die Brennpunktseigenschaften der sphärischen Ellipse und ihre Übertragung auf die ebene nichteuklidische elliptische Geometrie.** Math. Ann. **121**, 405–414 (1950).

A spherical ellipse is defined as either circuit of the intersection of a sphere (centre at the origin) with the cone $x^2\cot^2\alpha+y^2\cot^2\beta=z^2$ $(\pi/2>\alpha>\beta)$. It is shown that the curve has two internal points on the sphere, called its foci, given by $y=0$, $x=z\tan\gamma$ $(\cos\gamma=\cos\alpha/\cos\beta)$ such that the curve is the locus of a point on the sphere, the sum of whose spherical distances from the foci is $2\alpha$. The corresponding theorem in (elliptic) non-Euclidean geometry is stated.

*P. Du Val* (Athens, Ga.).

**Schilling, Friedrich. Die Brennpunktseigenschaften der eigentlichen Ellipse in der ebenen nichteuklidischen hyperbolischen Geometrie.** Math. Ann. **121**, 415–426 (1950).

A proper ellipse in hyperbolic non-Euclidean geometry is defined as a conic wholly interior to the absolute conic. It is shown that every proper ellipse has a centre of symmetry and two mutually perpendicular axes of symmetry, and two points (foci) in the major axis such that the ellipse is the locus of a point, the sum of whose distances from the foci is constant.

*P. Du Val* (Athens, Ga.).

**Timpanaro, Seb. Le interpretazioni della geometria non euclidea.** Boll. Un. Mat. Ital. (3) **5**, 82–85 (1950).

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## *Convex Domains, Integral Geometry, Extremal Problems*

**Levi, F. W. On the isoperimetric problem.** Math. Student **17** (1949), 1–25 (1950).

Expository article.

**Trost, E. Berichtigung zu der Note: "Über eine Extremalaufgabe."** Nieuw Arch. Wiskunde (2) **23**, 131–132 (1950).

[The paper quoted in the title appeared in the same vol., 1–3 (1949); these Rev. **10**, 395.] Let $c$ be a convex arc of a curve joining the points $A$ and $B$ and lying in the triangle $ACB$, $A^*$ and $B^*$ the points of intersection of $AC$ and $BC$ with a tangent of $c$ touching $c$ at $P$ and $r$ and $r_A$ the radii of curvature at $P$ of $c$ and the hyperbola passing through $P$ and having the asymptotes $AC$ and $BC$. In order for the area of the triangle $A^*CB^*$ to have an extremum it is necessary that $A^*P=PB^*$. It is shown by means of analytical tools that the inequality $r>r_A$ involves a maximum, whence $r<r_A$ a minimum.

*L. Fejes Tóth* (Veszprém).

**Sarantopoulos, Spyridion. Sur les courbes ayant la même longueur et la même (ou inférieure) courbure avec une autre courbe plane et convexe.** Bull. Soc. Math. Grèce **24**, 51–67 (1949). (Greek. French summary)

Theorems of A. Schur and E. Schmidt for the case of convex curves are extended to the case where the curve is locally convex. Let $p(s)$ and $p^*(s)$ be two curves in the plane with the same length and a finite number of corners for the same values of $s$. If the curvatures $k(s)$ and $k^*(s)$ of $p$ and $p^*$ satisfy the relation $0\leqq k^*(s)\leqq k(s)$ and if the corners are such that $p$ always turns in the same direction, and the angles between the positive tangents of $p^*$ are almost as large as the angles at the corresponding corners of $p$, then $|\delta^*|\cos(x^*,\delta^*)\geqq|\delta|\cos(\alpha,\delta)$, where $\delta$ $(\delta^*)$ is the vector leading from the initial point of $p$ $(p^*)$ to the end point of $p$ $(p^*)$ and $x, x^*$ are the positive tangents of $p$ and $p^*$ at corresponding points.

*H. Busemann.*

**Hadwiger, H. Elementare Ermittlung extremaler Rotationskörper.** Revista Mat. Hisp.-Amer. (4) **9**, 59–70 (1949).

The author considers convex bodies of revolution in the ordinary space subject to one or both of the following restrictions: the maximal parallel has a given radius $a\geqq 0$, the radii of the two circular disks bounding the body are greater than or equal to a given number $b\geqq 0$. For each of these classes the isoperimetric problem is solved: to find among all bodies of the class with a given volume that with minimum area of the total or curved surface. The solutions are spherical lenses, or bodies consisting of a circular cylinder and two hemispheres, or bodies obtained from these by truncating. The method is elementary but the classical isoperimetric inequality is used. Some of the results are known.

*W. Fenchel* (Princeton, N. J.).

**Aleksandrov, A. D., and Pogorelov, A. V. Uniqueness of convex surfaces of revolution.** Mat. Sbornik N.S. **26(68)**, 183–204 (1950). (Russian)

Let an unbounded convex surface in $E^3$ mean a set homeomorphic to the plane which is the complete boundary of an unbounded convex set in $E^3$. All surfaces considered are of class $C^3$. A surface $S$ is said to be uniquely determined within a certain class $\Gamma$ if every surface in $\Gamma$ which is intrinsically isometric to $S$ is congruent to $S$. An unbounded convex surface of revolution whose spherical image has measure less than $2\pi$ is uniquely determined within the class of all surfaces with the same spherical image. An unbounded convex surface of revolution $z=f(x^2+y^2)$ whose spherical image has measure $2\pi$ is uniquely determined (within the class of all $C^3$ surfaces) if an $m>0$ exists such that $f'(t)>m$. The main tool is the following theorem on infinitesimal (isometric) deformations. Let the unbounded convex surface $z=f(x,y)$ undergo an infinitesimal deformation for which the velocity in the direction of the $z$-axis of the point $(x,y,f(x,y))$ is $\zeta(x,y)$. When $(x,y,f(x,y))$ tends on the surface to $\infty$ let $\zeta(x,y)(x^2+y^2)^{-\frac{1}{2}}\to 0$. Then the deformation reduces to a motion.

*H. Busemann* (Los Angeles, Calif.).

**Aleksandrov, A. D. Quasigeodesics.** Doklady Akad. Nauk SSSR (N.S.) **69**, 717–720 (1949). (Russian)

On a surface with bounded curvature the right and left integral curvatures $\tau_r(\lambda)$, $\tau_l(\lambda)$ exist for every curve $\Lambda$ which has at each point a definite direction backward and forward. Define the total geodesic curvature of $\Lambda$ as the least upper bound of $\frac{1}{2}\sum_k\{|\tau_r(\lambda_k)|+|\tau_l(\lambda_k)|-|\tau_r(\lambda_k)+\tau_l(\lambda_k)|\}$ over all decompositions of $\Lambda$ into a finite number of arcs $\lambda_k$. A quasigeodesic is a curve whose total geodesic curvature is zero. Every shortest arc is a quasigeodesic but not conversely. For convex surfaces this definition coincides with the previous definition of the author [see Intrinsic Geometry of Convex Surfaces, OGIZ, Moscow-Leningrad, 1948; these Rev. **10**, 619]. A quasigeodesic has (with an obvious definition) right and left integral geodesic curvatures of bounded variation and is rectifiable.

The sequence of completely additive set functions $\mu_n$ defined on a manifold $R$ is said to converge weakly to $\mu$ if for every continuous function which differs from zero only on a compact set, $\int_R f(x)\mu_n(dE)\to\int_R f(x)\mu(dE)$. On a surface $R$

let inner metrics $\rho_1, \rho_2, \cdots$ of bounded curvature be defined. Then $\rho_n$ is said to tend regularly to $\rho_0$ if $\rho_n$ tends uniformly to $\rho_0$ on every compact subset of $R$, and if the positive and negative parts of the integral (Gauss) curvatures of $\rho_n$ tend weakly to those of $\rho_0$. For any surface of bounded curvature and metric $\rho_0$ there exist sequences of polyhedral or Riemannian metrics $\rho_n$ which tends regularly to $\rho_0$. If under regular convergence of $\rho_n$ to $\rho_0$ a sequence of quasigeodesics $\lambda_n$ of $\rho_n$ converges to a curve $\lambda_0$, then $\lambda_0$ is a quasigeodesic of $\rho_0$, and every quasigeodesic $\lambda_0$ of $\rho_0$ is the limit of a sequence of quasigeodesics $\lambda_n$ of $\rho_n$. On surfaces where the ratio of integral curvature to area is bounded, every quasigeodesic is a geodesic.

*H. Busemann* (Los Angeles, Calif.).

**Zalgaller, V. A. On curves with curvature of bounded variation on a convex surface.** Mat. Sbornik N.S. **26(68)**, 205–214 (1950). (Russian)

The variation of the right integral geodesic curvature $\tau_r(L)$ of a curve $L$ on a convex surface is defined as $\sigma_r(L)=\sup \sum_k |\tau_r(L_k)|$ over all decompositions of $L$ into a finite number of arcs $L_k$. If on a convex surface a Jordan arc $L$ with finite $\sigma_r(L)$ is given, then a sequence of arcs $L_n$ tending to $L$ from the right and having only the endpoints in common with $L$ exists such that $\limsup \sigma_r(L_n) \leqq \sigma_r(L)$. If two points are connected by a geodesic $L_0$ of length $\lambda_0$ and a curve $L$ of length $\lambda$ such that $\omega+\sigma_r(L)<\pi$ where $\omega$ is the integral curvature of the domain bounded by $L_0$ and $L$, then $\lambda \cos[(\sigma+\omega)/2] \leqq \lambda_0$. If a sequence of curves $L_n$ with $\sigma_r(L_n)<M$ tends to a curve $L$ with a finite number of multiple points, then the length of $L_n$ tends to the length of $L$.

*H. Busemann* (Los Angeles, Calif.).

**Rademacher, Hans, and Schoenberg, I. J. Helly's theorems on convex domains and Tchebycheff's approximation problem.** Canadian J. Math. **2**, 245–256 (1950).

A well-known theorem of Helly states that if a finite collection of convex sets in $n$-dimensional space has the property that every $n+1$ among the sets have a common point, then all sets have a common point. The authors give a very simple new proof of this theorem which they claim "to be the first proof to be entirely geometric, in the sense that every single one of its steps has an intuitive geometric meaning." A range of applications is made among which the following result may be mentioned. Let $f(x)$ be a real bounded function defined in $(a, b)$ and $P(x)$ the polynomial of degree not exceeding $n-1$ which minimizes the expression $\rho(x_0, x_1, \cdots, x_n)=\max_{0\leqq\nu\leqq n} |f(x_\nu)-P(x_\nu)|$, the $x_\nu$'s being given values of $(a, b)$. Consider $\sup_{x_\nu} \rho(x_0, x_1, \cdots, x_n)$ and the corresponding polynomial $P^*(x)$. Then $P^*(x)$ minimizes the expression $\sup_{a\leqq x\leqq b} |f(x)-P(x)|$.

*L. Fejes Tóth.*

***Vidal Abascal, E. Geometria Integral sobre las Superficies Curvas. [Integral Geometry on Curved Surfaces].** Publicaciones del Observatorio de Santiago, no. 7. Santiago de Compostela, 1950. 63 pp.

Consider a geodesically convex domain $D$ on a surface of class $C^3$, such that two points of $D$ lie on exactly one geodesic in $D$. Let $C$ be a closed curve of class $C^3$ and length $L$ in $D$, $\rho$ and $\varphi$ geodesic polar coordinates such that for each geodesic $G$ which intersects $C$ exactly one geodesic $\varphi=$constant normal to $G$ exists. Let $\omega$ be the variation in the sense of Levi-Civita of the tangent of $\rho=$constant. If $S$ is the set of points where the lines $\varphi=$constant are normal to the geodesic $G$ intersecting $C$, then $\iint_S d\omega d\rho = L$. This formula is quite similar to the corresponding formula in plane integral geometry, hence $dG=d\omega d\rho$ may be considered as a measure for the density of the $G$. If $G$ cuts a fixed curve $C'$ in a point $p$ where abscissa (=arc length) measured on $C'$ is $s$ and if $\psi$ is the angle formed by $G$ and $C'$ at $p$, then also $dG=\sin\varphi d\rho d\psi$. Let $a, b$ be two points, $a', b'$ neighboring points. In polar coordinates, with the intersection of the geodesic arcs through $a, b$ and through $a', b'$ as pole, let $da$ and $db$ be the changes of the radius vector corresponding to the transition from $a$ to $a'$ and from $b$ to $b'$. If $C$ is a closed convex curve, $a, b$ interior points of $C$, $F$ the area bounded by $C$, $\sigma$ the length of the arc intercepted by $C$ on $G$, $k$ the curvature of the surface, then Crofton's formula generalizes to

$$F^2=\int dG \int_0^\sigma \int_0^\sigma \sin[(a-b)k^1]\, da db - \int \left(2\frac{\partial k}{\partial \rho}\frac{\rho^4}{4!}+\cdots\right) dG da db.$$

If $k$ is constant, this reduces to

$$\int_{G\cap C\neq 0} \{\sin[\sigma k^1]-\sigma k^1\} k^1 dG = \tfrac{1}{2}(kF)^2.$$

Various other formulas for the length generalising known formulas for surfaces of constant curvature are given, for instance in terms of width. Some former results of the author [Bull. Amer. Math. Soc. **53**, 841–844 (1947); Revista Mat. Hisp.-Amer. (4) **7**, 132–142, 269–278 (1947); these Rev. **9**, 608] are obtained by new methods.

The second part of the paper shows how these ideas fit into the general theory of integral invariants by Poincaré and Cartan. The formalism is too complicated to be discussed here. The reading is simplified by the fact that the author derives Cartan's formulas again as far as they are necessary for his purpose.

*H. Busemann.*

**Santaló, L. A. Integral geometry in projective and affine spaces.** Ann. of Math. (2) **51**, 739–755 (1950).

This paper is an application to specific spaces of Chern's theory of integral geometry in a general Klein space [same Ann. (2) **43**, 178–189 (1942); these Rev. **3**, 253]. Let $\omega_1, \cdots, \omega_r$ be the relative components of a Lie group $G_r$ of automorphisms of a space $E$. If a continuous subgroup of $G_r$, $g_{r-k}$, leaves invariant the geometrical element $H$, the expression $[\omega_1, \omega_2, \cdots, \omega_k]$ is a density $dH$ for the set of elements $sH$ if and only if $[\omega_1, \omega_2, \cdots, \omega_k]'=0$. This condition is a modification of the condition given by Chern. By computing these densities the following results are obtained. In a projective space the linear subspaces have no invariant density. The elements $(S_{k_1}+S_{k_2}+\cdots+S_{k_m})$ composed of $m$ linear subspaces of dimensions $k_i$ without common point and such that $k_1+k_2+\cdots+k_m+m\leqq n+1$ have a density if and only if $k_1+k_2+\cdots+k_m+m=n+1$. In a unimodular center-affine space the linear subspaces through the origin have no density. The elements $(S_{k_1}+S_{k_2}+\cdots+S_{k_m})$ through the origin, and with no other common point, have a density if and only if $k_1+k_2+\cdots+k_m=n$. The points and general hyperplanes are the only linear subspaces which have a density. These results are applied to prove a theorem of Minkowski and Hlawka [Hlawka, Math. Z. **49**, 285–312 (1943); these Rev. **5**, 201]. In a unimodular affine space the points are the only linear subspaces which have a density. The elements $(S_{k_1}+S_{k_2}+\cdots+S_{k_m})$ with $k_1+k_2+\cdots+k_m+m\leqq n+1$

admit a density if and only if all $h_i=0$ or

$$h_1+\cdots+h_m+m=n+1.$$

*C. B. Allendoerfer* (Haverford, Pa.).

**Hopfner, F. Das Problem des bestanschliessenden Ellipsoids in der Geodäsie.** Akad. Wiss. Wien, S.-B. IIa. 156, 75–86 (1948).

It is shown that none of the infinite set of ellipsoids approximating the geoid satisfies the condition of best fit, namely that the sum of the squares of the angles between normals at corresponding points of the ellipsoid and the geoid be a minimum. Such minimal surfaces must be of a more general class. Conditions determining them are set up.

*N. A. Hall* (Minneapolis, Minn.).

**Haupt, O. Errata: Vierscheitelsatz und Umkehrung.** Ann. Mat. Pura Appl. (4) **28**, 345 (1949).

Cf. the same Ann. (4) **27**, 293–320 (1948); these Rev. **11**, 127.

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## *Algebraic Geometry*

**Bagchi, Haridas. Some novel properties of cyclides and hyper-cyclides.** J. Roy. Asiatic Soc. Bengal. Sci. **15**, 121–130 (1949).

For a general cyclide the five centres of inversion and the feet of the sets of normals from these to the related focal quadrics lie on a rectangular twisted cubic. The corresponding generalization for hypercyclides in $[n]$ is also verified.

*J. G. Semple* (London).

**Bagchi, Haridas. Note on the sextactic points of a cubic and its Hessian.** J. Roy. Asiatic Soc. Bengal. Sci. **15**, 131–138 (1949).

This paper gives an account of various geometrical relations connected with two triads of sextactic points, of a non-singular cubic and its Hessian respectively, which lie on one of the nine common harmonic polars of the two curves.

*J. G. Semple* (London).

**Hohenberg, Fritz. Die Haupttangentenkurven der Müllerschen Fläche.** Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 1949, ~~no. 14, 4 pp.~~ (1949).

Die Müllersche Fläche ist eine metrisch spezielle Fläche dritter Ordnung mit zwei konischen und einem biplanaren Knotenpunkt. In einem geeigneten rechtwinkligen Koordinatensystem hat sie die Gleichung $(x^2+y^2)z-c^2x=0$. Der Verf. bestimmt die Haupttangentenkurven mittels darstellender Geometrie; ihre Grundrisse sind Kardioiden, die Kurven selbst sind Raumkurven 5. Ordnung mit einer gemeinsamen Asymptote.

*O. Bottema* (Delft).

**Pompilj, Giuseppe. Per la caratterizzazione delle curve di diramazione dei piani quadrupli.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 77–93 (1949).

***Lockhart, Brooks Javins. Covariant Correspondences and Covariant Sets of Points Defined by a Given Correspondence on an Algebraic Curve.** Abstract of a Thesis, University of Illinois, 1943. ii+9 pp.

After some general discussion of valency correspondences on an algebraic curve and types of covariant relation which such correspondences may satisfy, the author obtains formulae for the number of cyclic $n$-ads of a valency correspondence, distinguishing between the cases when the correspondence is unsymmetric and symmetric. Applying one of these formulae to the quartic of genus 3 and to the correspondence between points of this curve whose join touches the curve elsewhere, he computes the numbers of contact curves of the quartic, of the various systems known to exist, which degenerate into proper polygons inscribed in and circumscribed to the curve.

*J. G. Semple.*

**Mathieu, Paulette. Extension de la correspondance involutive de Reye dans l'espace à cinq dimensions.** C. R. Acad. Sci. Paris **230**, 1567–1569 (1950).

The author continues the investigation begun in a former note [same vol., 1132–1134 (1950); these Rev. **11**, 614] of the involution in $S_5$ determined by quadrics $|Q|$ through a canonical curve $C^{10}$ of genus 6. (These are a linear system of dimension 5, in which a general linear subsystem of dimensions 4 have just a pair of points in common, besides the base curve, these being a pair of the involution.) The main result of the present note is that the homaloidal system of the Cremona transformation which interchanges the pairs of the involution is of order 11 with the following base elements: (1) the locus $V_3^{30}$ of chords of $C^{10}$, simple; (2) the five rational cubics $W_3^3$, each generated by planes joining sets of an involution of order 4 on $C^{10}$, double; (3) the unique quintic del Pezzo surface, of which $C^{10}$ is a quadric section, quintuple; (4) $C^{10}$ itself, sextuple. Characteristic threefolds, surfaces, and chords of this system are also studied, as well as the corresponding loci on the double $S_5$, projective model of $|Q|$.

*P. Du Val* (Athens, Ga.).

**Casulleras, Juan. Study of the correspondence between the Grassmannian of the lines of $E_3$ and the Segre variety which is the product of two planes.** Revista Mat. Hisp.-Amer. (4) **9**, 234–237 (1949). (Spanish)

Les droites de $E_3$ peuvent se représenter, d'une part par une quadrique de $E_5$ (grassmannienne), et d'autre part par les couples de points où elles coupent deux plans distincts de $E_3$. Ces derniers couples étant en correspondance biunivoque avec une variété de Segre de $E_8$, on a une correspondance entre cette variété et la grassmannienne de $E_5$. L'auteur étudie cette correspondance, ses points exceptionnels, et son comportement au voisinage de ces derniers points.

*P. Vincensini* (Marseille).

***Lapsley, Janie Campbell. Apolar Systems of Bilinear Forms.** Abstract of a Thesis, University of Illinois, 1943. ii+11 pp.

This thesis is a study of triply infinite and quadruply infinite linear systems of correlations on a pair of planes $S_2$ and $S_2'$, the geometry of the former being related to that of the latter by the use of the apolarity relation. The correlations of a triply infinite system being mapped by the points of a space $S_3$, the map of singular correlations of the system is a cubic surface; also the singular points of such singular correlations generate a quintic Cremona transformation $T$ of $S_2$ on $S_2'$ for which equations are here obtained in symbolic notation. Similarly, when the correlations of a quadruply infinite system are mapped by points of $S_4$, its singular members are mapped by a determinantal cubic primal $M$ of $S_4$, whose six nodes map correlations of rank 1. A number of known properties of $M$, including its generation by a net of rational normal quartic curves through the

nodes, are interpreted in terms of correlations and expressed algebraically. Representations of the points of $M$ on the first order line elements of $S_2$ and $S_2'$ are set up, and from these is derived a contact transformation, mapping the line elements of either plane on the other. It is shown that for any quintic Cremona transformation, such as $T$, with six fundamental points, there exists an associated contact transformation; this is obtained by associating with any quintic of the homaloidal net in $S_2'$, for example, its unique perspective conic, i.e., the unique conic which can be related parametrically to the quintic in such a way that a point of the latter and the related tangent to the former are incident. The last section discusses the three line systems on $M$ in relation to the plane sets into which the nodes project from lines of these systems. *J. G. Semple* (London).

***Strobel, Charles Frederick. The Quadrilinear Form (1, 1, 1, 2).** Abstract of a Thesis, University of Illinois, 1941. ii+5 pp.

This thesis deals with the geometry of a doubly infinite linear system of trilinear binary relations of the form $(\alpha r)(\beta s)(\gamma t)=0$, such a system being representable by the vanishing of a form $(\alpha r)(\beta s)(\gamma t)(\delta x)$ where $x$ is a ternary variable and the form is defined only to within an arbitrary constant factor. This geometry is related in turn to that of (a) a plane quartic curve $R$ with three of its contact systems of conics (mutually azygetic) individually isolated, (b) an associated linear net of quadrics, and (c) a linear net of cubic surfaces with three base lines, each surface of the net mapping the triads $(r, s, t)$ of one of the trilinear relations. There are, in general, six neutral triads $(r, s, t)$, satisfying all the relations of the system; these are such that five of them determine the sixth, and the six pairs of values of $r$ and $s$ determine the associated values of $t$. The general plane quartic curve $R$ can be associated with each of 2016 essentially distinct quadrilinear forms. *J. G. Semple.*

**Nollet, Louis. Sur la classification et la détermination des congruences linéaires de cubiques gauches.** Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **23**, no. 6, 112 pp. (1949).

The object of the work is the classification of congruences $G$ of twisted cubics in ordinary space which are linear in the sense that one curve of the congruence passes through a general point of space. A point common to more than one curve is either (a) on a "singular" curve, through whose general point pass $\infty^1$ curves of $G$, or (b) on an "exceptional" curve, which is a constituent of $\infty^1$ degenerate curves of $G$, or (c) a "principal" point, common to all curves of $G$. The congruence consists of all cubics which pass through the principal points and meet each singular curve a specified number of times. The first 20 pages are devoted to proving the following theorem. If $G$ has a singular curve $\gamma$ along which the curves of $G$ do not all touch the same surface, and to which no curve or part of a curve of $G$ is infinitely near, there exists either (i) a net of rational surfaces with $\gamma$ as either a simple or double base curve, and with plane sections of genus at most 5, whose variable intersection consists of one curve of $G$; or (ii) a pencil of rational surfaces of which $\gamma$ is either a simple base curve or not a base curve at all, and with plane sections of genus at most 5, each generated by $\infty^1$ curves of $G$. This gives a classification of congruences $G$ into twelve classes indicated by $G^{(i)}$ $(i=1, \cdots, 12)$; $G^{(2r+1)}$ indicates one for which the system given by the above theorem is a net, $G^{(2r+2)}$ one for which it is a pencil, in each case of surfaces of section genus $r$ $(r=0, \cdots, 5)$. It does not, however, seem to be the case that the system of surfaces on which this classification is based is unique, i.e., that another singular curve of the same congruence $G$ may not lead to another net or pencil of surfaces belonging to the congruence, of a different section genus. In fact one example is given in which the curves of $G$ are the variable intersections of surfaces of two pencils $|\Phi|$, $|\Psi|$, of which the former are sextics with a triple twisted cubic, a double line and five simple lines as base elements, while the latter are quintics with the cubic and the first line as double the other lines and two conics as simple base elements, so that their section genera are 0, 4; thus the congruence is at once a $G^{(3)}$ and a $G^{(10)}$.

The rest of the work is devoted to the complete listing of congruences in the first six classes, and to copious examples out of the remaining six. The number of distinct cases appears to be enormous. The actual results are scattered every few pages through the book, but are identifiable by the use of italics. Nothing approaching a summary of these is possible within the scope of this review. *P. Du Val.*

**Jongmans, F., et Nollet, L. Classification des systèmes linéaires de courbes algébriques planes de genre trois.** Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 6, 48 pp. (1949).

A partial classification is given of linear systems

$$C^t(M_1^{r_1}, \cdots, M_k^{r_k}), \qquad r_1 \geqq \cdots \geqq r_k,$$

(i.e., of curves of order $t$ with base points $M_1, \cdots, M_k$ of multiplicities $r_1, \cdots, r_k$, respectively) which satisfy the following conditions: (a) the general curve is of genus 3; (b) the system contains at least one actual irreducible curve; (c) the system is minimal, i.e., not equivalent to one of lower order under Cremona transformation, either because the three highest base multiplicities satisfy Nöther's inequality $r_1+r_2+r_3 \leqq t$ (species 1) or because $M_2$, $M_3$ are in the neighborhood of $M_1$ (species 2). First a purely arithmetical discussion is given which enumerates all virtual systems of grade $D \geqq 0$, satisfying (a), (c), but gives no information about (b). Clearly the freedom $r \geqq D-2$, equality holding if the base points are general in position. The cases $D \geqq 4$ are well known ($D=4$, $r=3$ being that in which the base points lie on a cubic and any one of them is determined in a finite number of ways by the rest).

The larger, and properly geometrical part, of the paper is devoted to picking out as far as possible the cases in which the base points can be chosen so as to give either $D=2$, $r=2$ (a net of hyperelliptic curves) or $D=0$, $r=1$ (a pencil with all its base points specified). The method depends on an analysis of the properties of the adjoint system. The classification arrived at is as follows. (A) Systems with irreducible adjoint system of genus $<3$: $C^4$ $(D=16, r=14)$; $C^6(7M^2)$ $(D=8, r=6)$; $C^7(M^3, 9N^2)$ $(D=4, r=3)$; $C^8(M^4, 12N^2)$ $(D=0, r=1)$ (two possible arrangements of base points; pencil compounded with de Jonquières' involution); $C^9(8M^3, N^2, P^1)$ $(D=4, r=3)$; $C^9(7M^3, 4N^2)$ $(D=2, r=2)$ (net compounded with Geiser involution); $C^{12}(8M^4, 4N^2)$ $(D=2, r=1)$ (pencil compounded with Bertini involution). Also of course systems obtained from these by adding simple base points.

(B) Systems whose adjoint system is compounded with a pencil of lines:

$$C^5(M^3); \quad C^6(M^4, N^2); \quad C^7(M^5, 2N^2); \quad C^8(M^6, 3N^2),$$

all of which have $D=16$, $r=14$; the two last are only mini-

mal if all $N$'s are in the neighborhood of $M$. Simple base points can be added to the first three, but the second ceases to be minimal unless $N$ is in the neighborhood of $M$, and the second and third cease to be minimal unless the added simple base points are consecutive on branches through $M$, and either are limited in number or the branches are inflexional.

(C) Nets ($D=2, r=2$) whose adjoint system is irreducible of genus at least 3:

$$C^9(7M^3, 4N^3);\quad C^{12}(8M^4, N^3, P^2, \bar{N}^1);\quad C^{15}(7M^5, 3N^4)$$

compounded with Geiser, Bertini and Geiser involutions, respectively, determined by the base points $M$, to which the other base points have special relations. Two other arithmetic systems, $C^{10}(M^4, 9M^3, P^1)$ and $C^{13}(8M^6, N^5, P^2)$, give $D=2$, but the base points cannot be chosen to make $r=2$.

(D) Pencils whose adjoint system is of genus at least 3. There are 20 arithmetically possible systems with $D=0$ or $D=1$ (leaving one simple base point to be determined by the rest); but it is not determined in which of these cases the base points can be chosen so as to give a pencil or even an actual curve.

(E) Isolated curves whose adjacent system is of genus at least 3 or is compounded with a pencil of elliptic curves. These remain to be found. *P. Du Val* (Athens, Ga.).

**Jongmans, François. Mémoire sur les surfaces et les variétés algébriques à courbes-sections de genre quatre.** Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **23**, no. 4, 95 pp. (1949).

Chapter I (nearly half the paper) is devoted to the classification of regular surfaces $F^m$ whose hyperplane sections are nonhyperelliptic curves of genus 4. The results were all anticipated, partly by Roth [Proc. Cambridge Philos. Soc. **29**, 184–194 (1933)] and partly by Du Val [Proc. London Math. Soc. (2) **35**, 1–13 (1933); J. London Math. Soc. **9**, 105–112 (1934)]. The author refers to the former, but the work is actually independent. Here $F^m$ is either (a) Enriques' sextic in $S_3$ with six double lines, the intersections by pairs of four planes ($p_0=p_a=0$, $P_2=1$, $m=6$); (b) the intersection of a cubic and a quadric in $S_4$, or the projection of this from a point into $S_3$ (all genera $=1$, $m=6$ or 5); (c) one of six types of rational surfaces, in $S_{m-3}$ save for special position of the base points which give $F^6$ in $S_4$ or $F^5$ in $S_3$; the mapping systems of plane curves are the following: (I) $|H^6(2A^3, (18-m)B)|$, i.e., sextics with two triple and $18-m$ simple base points, $5\leqq m\leqq 18$; (II) $|H^6(6A^3, (12-m)B)|$, $5\leqq m\leqq 12$; (III) $|H^7(A^3, 8B^2, (8-m)C)|$, $5\leqq m\leqq 8$; (IV) $|H^8(8A^3, (9-m)B)|$, $5\leqq m\leqq 9$; (V) $|H^9(7A^3, 3B^2)|$, $m=6$; (VI) $|H^{12}(8A^4, B^3, C)|$, $m=6$. Chapter II is devoted to the threefold varieties $M_3^m$ whose general hyperplane sections are these various types of $F^m$. Apart from cones they are as follows. (a) Sextic hypersurface in $S_4$ with six double planes, intersections by pairs of four $S_2$'s; the point of intersection of all these is generally quadruple, and it is not known whether the variety is rational, though all its genera are zero. (b) Intersection of a cubic and a quadric in $S_5$, or the projection of this from a point into $S_4$; generally irrational, though again all its genera are zero. (c) Rational $M_3^m$ with hyperplane sections of either of the six types listed above. The order $N$ and base elements are as follows: (I) (i) ($M_3^m$ ruled) $N=6$. 5-ple point $O$, two triple lines $a_1$, $a_2$, simple $(18-m)$-ic curve passing $(14-m)$-ply through $O$, and bisecant to $a_1$, $a_2$. (ii) Possibly other systems for $m\leqq 9$. (II) (i) ($M_3^m$ ruled) $N=6$. 5-ple point $O$, 6 double lines $a_i$ through $O$, simple $(12-m)$-ic curve passing $(7-m)$-ply through $O$ and bisecant to $a_i$. (ii) ($M_3^m$ nonruled) $N=3$. Simple plane cubic curve, $12-m$ simple points. (iii) Possibly other systems for $m\leqq 9$. (III) ($M_3^m$ nonruled) $N=4$. Triple line, 3 simple conics in planes through this. (IV) (i) ($M_3^6$ ruled, with double directrix line) $N=10$. 9-ple point $O$ with fixed cubic constituent $V^3$ of tangent cone: 8 triple lines $a_i$, simple line $b$ (locus of contact with $V^3$), 3 coplanar simple lines $c_1$, $c_2$, $c_3$, accidental simple line $m$, all generators of $V^3$; one simple line $p$ not passing through $O$, but coplanar with $c_1$, $c_2$, $c_3$. ($M_3^5$ ruled, with triple directrix line, projection of $M_3^6$ from a point of itself.) As for $M_3^6$, save that $b$ is double line. (ii) ($M_3^m$ nonruled) $N=6$. 4-ple point $P$ with plane $\alpha$ double constituent of tangent cone; tacnodal line $d$ through $P$ in $\alpha$, with fixed tangent plane $\beta$; three simple lines through $P$ in $\alpha$; simple sextic curve of genus 4 lying on a quadric cone which touches $\beta$ along $d$; $9-m$ simple points. (V) (i) ($M_3^6$ monoid in $S_4$) $N=6$. 3 coplanar double lines $a_1$, $a_2$, $a_3$, double twisted cubic $D$ through the intersections of these, simple elliptic sextic curve lying on a ruled quartic which has $a_1$, $a_2$, $a_3$ as generators and $D$ as double curve. (ii) ($M_3^6$ nonruled in $S_5$) $N=4$. Tacnode $P$ with fixed tangent plane $\alpha$, three torsal lines through $P$ in $\alpha$ with fixed tangent planes, a simple plane section through $P$. (VI) ($M_3^6$ nonruled in $S_6$) $N=4$. Unode $P$, tacnode $P'$ in the neighborhood of $P$, torsal line $p=PP'$ with fixed tangent plane $\pi$, simple conic in $\pi$ touching $p$ in $P$, simple sextic curve with tacnode in $P$ touching $p$ and both branches osculating $\pi$.

One or two of these varieties, notably (a), (b), (c), (IV) (ii), were studied by Fano [Ann. Mat. Pura Appl. (3) **24**, 49–88 (1915)]. The rest appear to be new. *P. Du Val.*

***Piazzolla Beloch, Margherita. Teoria diametrale delle curve algebriche piane.** Istituto di Matematica dell' Università di Ferrara, 1949. 95 pp. [Paged 1–82+ 1–13.]

A diameter of a plane curve of order $n$, conjugate to a direction $m$ (i.e., the direction $y=mx$) is defined as the locus of mean centres of the intersections of the curve with a variable line parallel to the direction $m$. (It is in fact simply the polar line of the point at infinity.) The condition for all diameters to pass through a point (called a principal point) is that the terms of order $n-1$ in the equation of the curve are linearly dependent on the partial derivatives of those of order $n$, the coefficients of these derivatives being $n$ times the coordinates of the principal point. The condition for all diameters to be parallel is that the terms of order $n$ are the $n$th power of a linear function. The condition for all diameters to coincide is that the terms of order $n$ are the $n$th power and those of order $n-1$ the $(n-1)$th power of the same linear function. A diameter perpendicular to the direction to which it is conjugate is called a principal diameter. In general the curve has $n$ principal diameters; if $n$ is odd it cannot have more, but if $n$ is even and the terms of highest order are a power of $x^2+y^2$, all diameters are principal.

It is stated that an asymptote is a diameter conjugate to its own direction; what actually happens is that when the direction $m$ approaches that of an asymptote the limit of the conjugate diameter is the asymptote; but the metrical definition of a diameter breaks down in this case. Where there are parallel asymptotes the limit of the diameter is a line parallel to these, locus of the mean centre of the intersections of a general line with the parallel asymptotes. In

an axis of symmetry is necessarily a principal diameter if $n$ is even, but not, of course, conversely. For $n$ odd every axis of symmetry is perpendicular to the direction of an asymptote.



general the curve has $\frac{1}{2}n(n-3)+1$ pairs of diameters, each conjugate to the direction of the other. If $n$ is odd there cannot be more than this, but if $n$ is even and the terms of order $n$ are a power of a quadratic expression, every diameter belongs to such a pair.

The diametral curve of order $h$ conjugate to the direction $m$ is defined as the locus of a point such that the sum of the products $h$ at a time of its distances from the intersections of the curve with the line through it parallel to the direction $m$ is zero. (It is in fact the polar curve of order $h$ of the point at infinity.) If there are $h$ parallel asymptotes, these constitute the (limit of the) diametral curve of order $h$ conjugate to their own direction.

Elementary methods are given for finding a centre or axis of symmetry if the curve has any. (An axis of symmetry is necessarily a principal diameter, but not of course conversely.) A curve of order $2n$ with two $n$-ple points at infinity is called a generalized conic, hyperbolic, elliptic, or parabolic according as these points are real and distinct, conjugate imaginary, or coincident. There is a principal point, the diameters form an involution in the pencil with this vertex, of pairs each conjugate to the direction of the other; one such pair is perpendicular, i.e., these two diameters are conjugate. In the parabolic case all diameters are parallel.

*P. Du Val* (Athens, Ga.).

**Arvesen, Ole Peder. Quelques applications de l'addition géométrique des courbes et des surfaces algébriques.** Norske Vid. Selsk. Forh., Trondheim **22**, no. 35, 163–166 (1950).

In a previous note [Norske Vid. Selsk. Forh. **12**, 115–118 (1940); these Rev. **2**, 13] the author defined addition of two algebraic plane curves. The present paper illustrates by a number of examples the extension of this concept to three dimensional space. The tangential equation of the sum of two curves or surfaces can be written down in determinantal form by using a theorem of the reviewer [Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 229–236 (1947); these Rev. **8**, 499].

*D. E. Rutherford* (St. Andrews).

**Godeaux, Lucien. Sur la construction de surfaces algébriques dont le diviseur de Severi est quelconque.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 919–923 (1949).

Soit $p$ un nombre premier arbitraire et $V$ la véronésienne des courbes planes $C^p$ d'ordre $p$ dans $S_n(x_0, x_1, \cdots, x_n)$ où $n=p(p+3)/2$. Dans $S_{n+3}$ contenant $S_n$ et un plan non incident $Y(y_1, y_2, y_3)$, considérons la surface $F$ d'équations $M_i(y)=\varphi_i(x)$ où $M_i$ représente successivement les $n+1$ monômes d'ordre $p$ en $(y)$ et $\varphi_i$, $n+1$ formes d'ordre $p$ en $(x)$ linéairement indépendantes et sans zéro commun (non irrelevant). La surface $F$ est d'ordre $p^{n+1}$ et projetée de $Y$ sur $S_n$ suivant $\Phi$ transformée algébrique de $V$. L'homologie biaxiale périodique $H$ $x_k'/x_k=y_k'/ay_k$ où $a=\exp 2i\pi/p$ définit sur $F$ une involution $I_p$ sans points unis dont $\Phi$ est une image. Les $S_{n+2}$ de $S_{n+3}$ passant par $S_n$ ou par $Y$ découpent sur $F$ des courbes $C_1$ et $C_2$; les $V_{n+3}^{p-1}$ de $S_{n+3}$ se répartissent en $p$ systèmes linéaires globalement invariants dans $H$, qui découpent sur $F$ les systèmes $|D_0|, \cdots, |D_{p-1}|$ et $|D_k|$ contient les courbes formées de $k$ courbes $C_1$ et de $p-1-k$ courbes $C$. Les $p$ systèmes correspondants sur $\Phi$ sont distincts et leurs multiples par $p$ sont équivalents, de sorte que le diviseur de Severi de $\Phi$ est $p$. On ne connaissait jusqu'ici que peu d'exemples de surfaces dont le diviseur de Severi dépasse l'unité.

*L. Gauthier* (Nancy).

**Godeaux, Lucien. Sulla costruzione di certe superficie algebriche irregolari.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 694–696 (1949).

Soit $\Gamma$ une courbe canonique (non hyperelliptique) de genre $\pi$, contenant une involution cyclique $i_p$ de genre $\pi'$, d'ordre premier $p$, sans points unis. Soit $\Gamma'$ une image de $i_p$. La série canonique $|G|$ de $\Gamma$ contient $p$ séries partielles globalement invariantes dans $i_p$, découpées par les hyperplans passant par les axes d'une homographie $H$ de période $p$. Ces séries partielles ont pour images $|G_0'|, \cdots, |G_{p-1}'|$ d'ordre $2\pi'-2$; $|G_0'|$ a la dimension $\pi'-1$ et est la série canonique de $\Gamma'$; les $|G_k'|$, $k\neq 0$, ont la dimension $\pi'-2$ et sont paracanoniques. Soit $F$ la surface des couples de points non ordonnés de $\Gamma$; ses genres sont $p_g=\frac{1}{2}\pi(\pi-1)$ et $p_a=\frac{1}{2}\pi(\pi-3)$. La surface $F$ contient une involution cyclique $I_p$ sans points unis, obtenue en associant les couples de $\Gamma$ dans $i_p$. Soit $F'$ une image de l'involution $I_p$; $F'$ contient à son tour une involution $I_p'$ dont l'image est la surface $\Phi$ des couples de points non ordonnés de $\Gamma'$. Cette surface $\Phi$ a l'irrégularité $\pi'$ et l'auteur montre, en calculant les genres de $F'$, que $F'$ a également l'irrégularité $\pi'$. Si $|C'|$ est le système canonique de $F'$, il existe sur $F'$, $p-1$ autres systèmes de même dimension, qui ont le même multiple par $p$ que $|C'|$.

*L. Gauthier* (Nancy).

**Turri, Tullio. Osservazioni a ricerche sopra le tabelle dei periodi degli integrali abeliani reali.** Rend. Sem. Fac. Sci. Univ. Cagliari **17** (1947), 29–33 (1948).

In questa nota l'autore crede ravvisare alcune contraddizioni in un lavoro di A. Comessatti [Ann. Mat. Pura Appl. (4) **2**, 67–106 (1925); **3**, 27–71 (1925), in particolare, **3**, pg. 35 e sgg.]. Ciò appare ovviamente dovuto ad un malinteso. Infatti nella parte centrale del lavoro si asserisce che: data una curva reale di genere 2 corrispondente ad una tabella di periodi per gli integrali di prima specie della forma:

$$\begin{Vmatrix} 1 & 0 & i\tau_{11} & \frac{1}{2}+i\tau_{12} \\ 0 & 1 & \frac{1}{2}+i\tau_{21} & i\tau_{22} \end{Vmatrix}$$

colle $\tau_{ik}=\tau_{ki}$ reali e $\Delta=\tau_{11}\tau_{22}-\tau_{12}^2>0$, (a) possono esistere curve con $\Delta\neq\frac{1}{4}$ possedenti integrali ellittici; (b) possono esistere curve con $\Delta=\frac{1}{4}$ non possedenti integrali ellittici. L'affermazione (a) ovvia perchè, come è noto sin dalle prime ricerche di H. Poincaré sugli integrali riducibili, nell' "intorno" di ogni varietà abeliana esistono varietà abeliane a integrali tutti riducibili ad integrali ellittici. Nè ciò contraddice ad alcuno dei risultati di Comessatti. L'asserzione (b) è invece falsa dato che si possono risolvere (contrariamente a quanto dice l'autore) in numeri razionali le equazioni nelle coordinate $p_{ik}$ d'una retta, asse di un siffatto integrale ellittico, ponendo ad es.: $p_{14}=p_{23}=p_{34}=p_{13}=0$; $p_{12}=p_{42}=1$. Ciò in accordo col lavoro di Comessatti.

*F. Conforto* (Princeton, N. J.).

**Turri, Tullio. Sul genere delle curve gobbe di De Jonquières.** Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 32–37 (1949).

L'auteur complète et corrige des résultats obtenus à propos de ces courbes par Campedelli [Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 171–191 (1942); ces Rev. **8**, 221]. Soit $T$ une transformation de de Jonquières entre plans superposés; l'intersection d'un faisceau de droites de sommet $P$ et des transformées par $T$ définit la courbe isologique de sommet $P$. Il existe deux telles courbes transformées l'une de l'autre par $T^2$, donc birationnellement identiques. Si l'on considère une droite par $P$ extérieure au plan et sur celle-ci deux points $S$ et $S'$, les droites $SM$ et

$S'M'$ joignant deux points homologues de ces deux courbes isologiques se coupent selon une courbe gauche de de Jonquières; si elle est irréductible, elle est de genre $n-1$ ou $n-2$, $n$ étant l'ordre de $T$. Cette courbe appartient à une quadrique; si celle-ci n'est pas un cône, la courbe de de Jonquières coupe un système de génératrices en deux points, l'autre en $n$ points. Si c'est un cône, les génératrices sont bisécantes de la courbe de de Jonquières. *B. d'Orgeval.*

**Turri, Tullio. Sulle trasformazioni piane cicliche di De Jonquières.** Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 38–43 (1949).

Une transformation plane cyclique de de Jonquières de période supérieure à 2 peut toujours par transformations birationnelles se ramener à une transformation possédant un faisceau de droites unies. S'il n'en est pas ainsi, la considération de la surface rationnelle image d'un groupe cyclique du plan permet d'obtenir le résultat. Si on choisit le sommet du faisceau au point à l'infini de l'axe $Oy$, on peut mettre cette transformation sous la forme simple: $x'=x$, $y'=[(\xi+1)\theta\psi y+(\xi-1)\psi^2]/[(\xi-1)\theta^2 y+(\xi+1)\theta\psi]$, $\theta$ et $\psi$ étant les polynômes de degré $h$ et $h+1$ en $x$, $\xi$ la racine $p$ème de l'unité. La courbe des points unis se décompose en deux courbes rationnelles: $\theta y+\psi=0$, $\theta y-\psi=0$.
*B. d'Orgeval* (Grenoble).

**Dedò, Modesto. Sulle trasformazioni di De Jonquières.** Boll. Un. Mat. Ital. (3) **4**, 353–359 (1949).

The author is concerned with de Jonquières transformations $J$ in a plane, of order $n$, such that the $(n-1)$-fold points $O, O'$ of the curves which are the images, under $J$ or $J^{-1}$, of the lines of the plane coincide. Such a transformation defines a projectivity $\omega_h$ between pairs $h, h'$ of corresponding lines through the multiple point. By forming the product of $J$ with a suitable projectivity, it may be assumed that $h$ and $h'$ coincide. The transformation is then completely determined if (a) three pairs of corresponding curves unisecant to the pencil centre $O$ are given, (b) one pair of corresponding unisecant curves and a united bisecant curve are given, (c) the united bisecant curve is given, and the transformation is involutory. In each case the order of $J$ and its fundamental points are determined. The main interest of the investigation is the reduction in the order of $J$ which occurs if the intersections of the given corresponding curves assume special positions in the plane. Thus, for instance, in case (a) if $a_1, a_1'$; $a_2, a_2'$; $a_3, a_3'$ are the three pairs of corresponding curves, and if a pair of intersections of $a_2, a_3$; $a_2', a_3'$ (other than $O$) are collinear with $O$, the order is reduced by one; if $a_1, a_2, a_3$ have a common point (with distinct tangents) outside $O$, the order is reduced by one; if both these conditions hold, the order is still only reduced by one; while if $a_1, a_2, a_3$ have a common point $F$ and $a_1', a_2', a_3'$ have a common point $G$ lying on $OF$ the order is in general reduced by two, but is reduced by three if the pencils $ha_1a_2a_3$, $ha_1'a_2'a_3'$, where $h$ is $OFG$, have the same cross-ratio. *J. A. Todd* (Cambridge, England).

***Freudenthal, Hans. La géométrie énumérative.** Topologie algébrique, pp. 17–33. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

The author begins with an exposition of some fundamental ideas of enumerative geometry, and goes on to show how the ideas of algebraic topology throw light on Schubert's procedures. In particular he uses a simplified form of a principle due to Pontrjagin [Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 389–422 (1939); these Rev. **1**, 259] to establish the base for conditions of dimension two on the lines of [3], and applies the "inverse homomorphism" of Hopf [J. Reine Angew. Math. **163**, 71–88 (1930)] to justify Schubert's results in some problems concerning complete conics (including a demonstration that $\eta\mu\rho^3$ is equal to 6, not 3). The paper concludes with a sketch of the author's enumerative treatment of cubic scrolls in [3], and gives the answer 504 to the problem (set by the Dutch Mathematical Society in 1942) of finding the number of such scrolls through 13 general points. *D. B. Scott* (London).

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## *Differential Geometry*

**Viguier, Gabriel. Les développantes généralisées du second ordre d'une courbe plane.** Experientia **6**, 134–135 (1950).

Special two-parameter families of involutes of a plane curve are defined, and their differential equations found.
*P. Franklin* (Cambridge, Mass.).

**Hoborski, A., et Gołąb, S. Sur les lignes de courbure spéciales.** Prace Mat.-Fiz. **47**, 17–20 (1949).

The authors consider a special kind of lines of curvature on an $X_2$ in $R_3$ and prove that every point of such a curve is an umbilic. *J. A. Schouten* (Epe).

**Marcus, Fred. Sulle superficie a curvatura totale costante.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 199–202 (1950).

Let two surfaces $S$ and $S'$ be in one-to-one point correspondence such that (a) their lines of curvature correspond, (b) a conjugate net on $S$ corresponds to a conjugate net on $S'$, and (c) the maximum and minimum radii of normal curvature of $S'$ correspond respectively to the minimum and maximum radii of normal curvature of $S$. It is proved on these hypotheses that the total curvature of $S$ (and therefore of $S'$) is a constant. *V. G. Grove* (East Lansing, Mich.).

**Löbell, Frank. Linienelementfunktionen und geodätische Ableitungen in der Flächentheorie.** Math. Ann. **121**, 427–445 (1950).

A vector or scalar function on a surface whose value depends on a point $P$ and a tangent direction at $P$ is called a line element function. The author here introduces a differentiation process which he calls geodesic differentiation which, when applied to a line element function, yields another such function. If the vectors $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{n}$ are a surface trihedral for an arbitrary curve on the surface, the vector function $\mathfrak{g}=T\mathfrak{t}+N\mathfrak{b}$ is clearly a line element function, where $T$ and $N$ are the geodesic torsion and normal curvature respectively in the direction $\mathfrak{t}$. By applying geodesic differentiation to $\mathfrak{g}$, new line element functions and position functions on the surface are derived. In terms of these functions the familiar Gauss and Codazzi-Mainardi equations are exhibited in a remarkably simple form. Numerous references to related work on line element functions and invariant differentiation are given in the paper. *S. B. Jackson.*

**Pinl, M. Abwickelbare Schiebflächen in $R_n$.** Comment. Math. Helv. **24**, 64–67 (1950).

It is proved that a translation surface [Schiebfläche] in $R_n$ is developable if and only if the transvection of two

simple trivectors (containing first and second derivatives only) vanishes. If $n=3$ this is only possible if one of the trivectors vanishes. This leads only to planes or cylinders. For $n>3$ the 3-directions of the trivectors (that have always a 2-direction in common) must be orthogonal. In this case the osculating planes of the two sets of curves of translation [Schiebkurven] span an $R_4$ at each point. An example is given in $R_4$. *J. A. Schouten* (Epe).

**Simonart, Fernand. Réseaux isothermes sur une surface.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 924–930 (1949).

The author [same vol., 346–356 (1949); these Rev. **11**, 52] showed that the net of plane curves $(Pdx+dy)(dx-Pdy)$ is an isothermally orthogonal net if

$$(1) \qquad (P_{xx}+P_{yy})/(P_x^2+P_y^2)=2P(1+P^2).$$

He extends this notion to curves $Pdu+dv=0$ on a surface $S$, showing that $(Pdu+dv)(du-Pdv)$ is isothermally orthogonal if $P$ satisfies (1) and the parametric curves $u=$constant, $v=$constant are isothermic. *V. G. Grove.*

**Lemoine, Simone. Sur les surfaces déformables avec persistance d'un réseau conjugué de courbes coniques.** C. R. Acad. Sci. Paris **230**, 920–922 (1950).

Let $S$ be a surface and $N$ a conjugate net on $S$. Let $S$ be deformed into $S'$ and $N$ into $N'$. Suppose $N'$ is a conjugate net on $S'$. The purpose of this note is to prove the following theorem. If the given net $N$ is composed of curves along which the developables circumscribing $S$ are cones, then the fundamental quadratic forms of $S$ may be expressed in terms of a single function $Q$ which satisfies two partial differential equations of the second order. *V. G. Grove.*

**Nadile, Antonio. Sulle reti incidenti alle reti di traslazione.** Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (4) **2(47)**, 72–78 (1945).

**Nadile, Antonio. Sulle congruenze coniugate alle reti $P$.** Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (4) **2(47)**, 79–88 (1945).

Let $S$ and $S'$ be surfaces in one-to-one point correspondence and $N$ and $N'$ two corresponding conjugate nets. The author says that $N$ and $N'$ are incident if the tangents to the corresponding curves of $N$ and $N'$ intersect. This relationship has previously been called the relation $F$ between $N$ and $N'$. The surface $S$ whose equations are $x^i=A^i(u^1)+B^i(u^2)$ is a surface of translation, the parametric net $N$ being a conjugate net. The direction numbers of a line $l$ generating a congruence $\Gamma$ conjugate to $N$ are given by

$$l^i=\int U^1A^i{}_{,\rho}du^\rho+\int U^2B^i{}_{,\rho}du^\rho,$$

wherein $U^1$, $U^2$ are functions of $u^1$ alone and $u^2$ alone. All of the nets conjugate to $\Gamma$ are now determined, and hence all of the nets incident (in relation $F$) to $N$ are found. The second paper determines congruences harmonic to the conjugate net $N$ described above on a surface of translation. Such a net $N$ is a special case of a "net $P$" defined as a conjugate net the homogeneous coordinates of whose points are solutions of $\partial^2\theta/\partial u^1\partial u^2=0$. All congruences conjugate and harmonic to a net $P$ are determined. *V. G. Grove.*

**Haack, Wolfgang. Strahlenkomplexe mit integrablen Pfaffschen Differentialformen.** Math. Z. **52**, 322–341 (1949).

Let $g$ be a generator of a complex $\Gamma$ of lines. Consider through $g$ a ruled surface $S$ whose generators belong to $\Gamma$. To $S$ the formula $\bar{\varphi}\tan\varphi=\delta-k$ of Koenig may be applied. The paper studies $\Gamma$ by studying the ruled surfaces $S$ of $\Gamma$. Three classes of surfaces arise according as (1) $\delta=k$, $\bar{\varphi}=0$; (2) $\bar{\varphi}=0$, $\varphi=\pi/2$; (3) $\delta=k$, $\varphi=0$. The author showed in a previous paper [Math. Z. **40**, 560–581 (1935)] that each of three classes of surfaces satisfies a differential equation of Pfaff. The question of the existence of nondegenerate complexes in which the Pfaffian forms are completely integrable is answered. Typical theorems may be stated as follows. If the Pfaffian form in case (1) is completely integrable then $\Gamma$ is composed of a pencil of isotropic congruences. A nondegenerate complex whose three Pfaffian differential forms are completely integrable may be represented by isotropic vectors which are functions of two arbitrary functions of two conjugate complex variables. A real complex with three complete integrable differential forms is determined by two arbitrary functions of a complex variable. *V. G. Grove.*

**Backes, Fernand. Sur une figure de référence mobile constituée par cinq sphères non nécessairement orthogonales.** C. R. Acad. Sci. Paris **230**, 1252–1253 (1950).

L'auteur indique les résultats qu'il a obtenus en substituant, dans la théorie du pentasphère mobile, au pentasphère orthogonal ordinairement employé, un pentasphère oblique quelconque. Les formules relatives au pentasphère fixe sont assez compliquées, mais celles relatives au pentasphère mobile différent très peu de celles qui concernent le cas où le pentasphère est orthogonal. Le carré de l'élément linéaire de l'espace se présente sous la forme $N\cdot\Omega(dx_1,\cdots,dx_5)$, $\Omega(x_1,\cdots,x_5)$ étant la relation quadratique liant les coordonnées pentasphèriques $x_i$ d'un point quelconque de l'espace. Comme dans le cas orthogonal une sphère a pour équation $\sum m_ix_i=0$. L'introduction des coordonnées métriques $m_i$, définies par la condition $\sum_i\sum_j m_im_j\cos V_{ij}=1$ (où $V_{ij}$ est l'angle des sphères $S_i$, $S_j$ du pentasphère fixe) simplifie certains résultats, entre autres l'expression de l'élément angulaire de deux sphères infiniment voisines, la condition de contact d'une sphère et d'une surface ou d'orthogonalité de deux surfaces, le lien entre les coordonnées pentasphèriques absolues $x_i$ et relatives $x_i$ d'un même point, et le développement de ces dernières au moyen des rotations du pentasphère mobile avec application à la caractéristique d'une surface dépendant d'un paramètre. Les formules relatives aux variations infinitésimales des sphères ou des cercles sont tout à fait analogues à celles concernant les points, et cela qu'il s'agisse d'un mouvement du pentasphère mobile à un ou deux paramètres.
*P. Vincensini* (Marseille).

***Eisenhart, Luther Pfahler. Riemannian Geometry.** 2d printing. Princeton University Press, Princeton, N. J., 1949. vii+306 pp. $3.50.

In this printing, errata in previous editions have been corrected in the text. Other revisions are in the appendix (37 pages), which also contains new material. There is an additional bibliography but the index does not include the appendix. From the new material we mention the expression of the metric tensor in terms of normal coordinates; the canonical forms of the metric tensor of a $V_n$ which admits $r$ independent fields of parallel vectors; the orthogonal coordinate systems of Stäckel; spaces $V_n$ of class greater than 1; the normal complexes of a $V_n$ in $R_m$; a theorem of T. Y. Thomas concerning a $V_n$, $n>3$, of class one; Einstein spaces of class one for $n\geqq 4$; the independent infinitesimal conformal transformations of a $V_3$ and of a conformally flat $V_n$, $n>3$; and the $V_n$ admitting a simply transitive group of motions. *J. A. Schouten* (Epe).

**Cossu, Aldo. Proprietà di curvatura di una particolare classe di varietà a connessione affine.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 702–707 (1949).

If a contravariant vector in an $A_n$ suffers a pseudo-parallel displacement along the border of an infinitesimal facet, the difference between the initial value and the final value is contained in the facet for every choice of the facet if and only if

$$(n-1)R^{\cdots\kappa}_{\nu\lambda\cdots}=A^{\kappa}_{\nu}R_{\rho\lambda}-A^{\kappa}_{\rho}R_{\nu\lambda}.$$

Bompiani has already proved that in a $V_n$ with this property the pseudoparallelism and the parallelism of Severi are identical. If the facet is changed the difference vectors belonging to a vector $v^\kappa$ always lie in the $E_{n-1}$ of the covariant vector $v^\kappa R_{\nu\lambda}$. The directions of the cone $R_{\nu\lambda}d\xi^\nu d\xi^\lambda=0$ are characterized by the property that they do not change with pseudoparallel displacement for all facets containing the direction. Many other properties are deduced, especially for the case that $R_{\nu\lambda}$ has the highest rank $n$ and for the case $n=3$. Remarkable properties arise if the facet lies in the cone. Covariant vectors can be treated in a similar way.

*J. A. Schouten* (Epe).

(Please see the note to p. 2401, by Tsuji.)

**Yano, Kentaro, Takano, Kazuo, and Tomonaga, Yasuro. On infinitesimal deformations of curves in spaces with linear connection.** Jap. J. Math. **19**, 433–477 (1948).

If a vector $v^*$ suffers a deformation, the value after the deformation can be compared with (1) the field value, (2) the parallel displaced vector, (3) the vector dragged along. This gives rise to three invariant variations $\overset{n}{\delta}v^k$, $\overset{t}{\delta}v^k$ and $\overset{a}{\delta}v^k$ whose commutators with $\nabla$ have the fundamental property that they are invariant if the variation of $v^k$ is changed [Schouten and van Kampen, Prace Mat.-Fiz. **41**, 1–19 (1934); Schouten and Struik, Einführung in die neueren Methoden der Differentialgeometrie, 2d ed., v. 1, Noordhoff, Groningen-Batavia, 1935, p. 148]. The most important operators are $\overset{t}{\delta}$ and $\overset{a}{\delta}$; $\overset{t}{\delta}$ was discovered previously by Hayden [Proc. London Math. Soc. (2) **32**, 321–336 (1931)]. Both operators (here called $\Delta$ and $D$) are used in this paper for a thorough investigation of the deformation of a curve in $L_n$. In the first four sections $\Delta$ and $D$ are defined and the commutation rules are established. In § 5 and § 6 the $\Delta$- and $D$-variations of the successive covariant derivatives of the tangent vector are computed. In § 7 and § 8, the same is done for the normals and curvatures. From now on special problems are considered: parallel tangent deformation, Combescure transformations, Bertrand curves in $V_n$, transformations carrying (a) geodesics into geodesics, (b) affine (projective) conics into affine (projective) conics, (c) Riemannian circles into Riemannian circles, (d) conformal circles into conformal circles. In all these sections the treatment is a double one, once by means of $\Delta$ and once by means of $D$.

*J. A. Schouten* (Epe).

**Yano, Kentaro. Sur la théorie des déformations infinitésimales.** J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 1–75 (1949).

This is an expository memoir on the theory of infinitesimal deformations which started with the publication of Levi-Civita's paper on geodesic deviation [Math. Ann. **97**, 291–320 (1926)]. The first extensions of the results of Levi-Civita on geodesic lines were carried out by Cartan [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **5**, 609–613 (1927)] for surfaces, by Bortolotti [Giorn. Mat. Battaglini **66**, 153–191 (1928)] for subspaces of a Riemannian space, and by Hayden [Proc. London Math. Soc. (2) **37**, 416–440 (1934)] for subspaces of a metric space with torsion. The study of deformation problems received an important impetus from the work of W. Slebodziński [Prace Mat.-Fiz. **39**, 55–62 (1932)] who generalized the symbol of an infinitesimal transformation to any tensor. This operator was called Lie derivation by Schouten and van Kampen [Prace Mat. Fiz. **41**, 1–19 (1933)] who showed how it could be exploited in the study of deformation problems in general. The Lie derivative is the main tool used throughout the paper under review, the general object of which is to relate and unify the various contributions which have been made to this field. The author frequently gives geometrical interpretations of the formulae appearing in the paper. In particular he gives a geometrical interpretation of the Lie derivative of the coefficients of affine connection.

*E. T. Davies* (Southampton).

**Walker, A. G. Canonical form for a Riemannian space with a parallel field of null planes.** Quart. J. Math., Oxford Ser. (2) **1**, 69–79 (1950).

This paper follows an earlier one by the author [same Quart., Oxford Ser. (1) **20**, 135–145 (1949); these Rev. **11**, 460]. An $r$-plane at a point $P$ of a Riemannian $V_n$ is an $r$-dimensional algebraic vector space ($r\leqq n$) whose elements are vectors at $P$, and the $r$-plane is null if all its elements are null vectors. A field of $r$-planes is parallel, and is called a parallel $r$-plane, if for any two connected points $P$, $Q$, a vector in the plane at $P$ is displaced into a vector in the plane at $Q$ by parallel transport along a curve from $P$ to $Q$. In the present paper the author obtains a canonical form valid in some neighborhood of $V_n$ in which there exists an allowable coordinate-system $(x^i)$, $V_n$ being assumed either of class $C^\omega$ or of class $C^\infty$. His main result is: A canonical form for the general $V_n$ admitting a parallel null $r$-plane is given by

$$(g_{ij})=\begin{bmatrix} O & O & I \\ O & A & H \\ I & H' & B \end{bmatrix}$$

where $I$ is the unit $r\times r$ matrix and $A$, $B$, $H$, $H'$ are matrix functions of the coordinates which are arbitrary but for the conditions (i) $A$, $B$, $H$ of orders $(n-2r)\times(n-2r)$, $r\times r$, $(n-2r)\times r$, respectively; $A$ nonsingular; $H'$ the transpose of $H$; (ii) $A, H, H'$ independent of the coordinates $x^1, x^2, \cdots, x^r$. A basis for the null $r$-plane is the set of vectors $\delta_1{}^i, \delta_2{}^i, \cdots, \delta_r{}^i$ at every point of $V_n$.

The proof of this theorem depends upon the lemma: Let $X_\alpha$ ($\alpha=1, 2, \cdots, m$) be independent linear differential operators satisfying $X_\alpha X_\beta - X_\beta X_\alpha = \Phi^\gamma{}_{\alpha\beta}X_\gamma$ for some $\Phi^\gamma{}_{\alpha\beta}$, and let $\phi_\alpha$ be given functions of the $x$'s. Then the system of equations $X_\alpha f=\phi_\alpha$ for $f$ admits a solution if, and only if, $X_\alpha\phi_\beta - X_\beta\phi_\alpha = \Phi^\gamma{}_{\alpha\beta}\phi_\alpha$. The paper concludes with a discussion of certain particular cases of the main theorem. The question whether the theorems of this paper have global implications is left by the author for future consideration.

*H. S. Ruse* (Leeds).

**Bompiani, E. Über das Theorem von F. Schur in der Riemannschen Geometrie.** Math. Z. **52**, 623–626 (1950).

This is a new proof of the theorem of Schur without recourse to the Bianchi identities but with strong use of parallel displacement. The simplicity of geometric meaning of the several lemmas used effectively removes any mystery that may still cling to this famous theorem. Along similar lines a proof is given of the theorem that a hypersurface in Euclidean space cannot have constant negative curvature.

*J. L. Vanderslice* (College Park, Md.).

**Bompiani, Enrico. Topologia differenziale. I. Enti topologici determinati da elementi differenziali di curve.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 3–8 (1950).

This is the first of a series of papers studying differential elements (of curves or varieties) at a point $O$ of a topological manifold (i.e., a space subject to general coordinate transformations). The differential elements are $k$-dimensional neighborhoods of $O$ of various differential orders (calottes). Designating a curvilinear calotte of order $s$ by $E_s$, the principal result of this note is the following: $k$ linearly independent and mutually tangent $E_2$'s determine an invariant first order calotte of $k$ dimensions, $\sigma$, called the principal subspace, and any $k$ dimensional calotte of second order containing the $k$ $E_2$'s must be tangent to $\sigma$.

*J. L. Vanderslice* (College Park, Md.).

**Bompiani, Enrico. Topologia differenziale. II. Invarianti topologici di elementi curvilinei.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 8–15 (1950).

In this second note [see the preceding review] the effect of changes of coordinates and calotte parametrization is considered. Various differential invariants are derived and their metric significance discussed. A typical such invariant is one depending on $k$ mutually tangent $E_2$'s and $k$ directions lying in their principal subspace.

*J. L. Vanderslice* (College Park, Md.).

**Bompiani, Enrico. Topologia differenziale. III. Calotte superficiali del 2° ordine tangenti in un punto.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 81–86 (1950).

[See the two preceding reviews.] The principal subspace of two tangent surface calottes of second order is defined. It contains their common tangent 2-space and three invariant vectors, not necessarily independent, thus can be of 3, 4, or 5 dimensions. These three cases are studied in order. For example, in the last case, given two $E_2$'s, one in each calotte and mutually tangent, the principal subspace of the $E_2$'s lies in a certain 3-space contained in the principal 5-space of the original calottes. Moreover, as the common tangent line varies, these 3-spaces describe a quadric cone which is the topological analogue of the cone of Del Pezzo.

*J. L. Vanderslice* (College Park, Md.).

***Bell, John Clarence. An Extension of the Parallelism of Clifford.** Abstract of a Thesis, University of Illinois, 1943. ii+17 pp.

Let $E_n$ denote the Cartesian $n$-dimensional space with the Euclidean metric $d\sigma=(d\mathbf{x}, d\mathbf{x})$, $R_n$ the Riemannian space having the same points except the origin $\mathbf{0}$ with the metric $ds=d\sigma/\rho$ where $\rho^2=(\mathbf{x}, \mathbf{x})$. For $\rho=$const. the metric in $R_n$ is the $(n-1)$-dimensional spherical metric. The geodesics in $R_n$ are (plane) equiangular spirals with center $\mathbf{0}$, the limiting cases being circles with center $\mathbf{0}$ and rays from $\mathbf{0}$. For two arbitrary points $x_1$ and $x_2$ determining with $\mathbf{0}$ a plane $P$, the distance $d$ measured along the geodesic corresponding to the determinations $(\rho_1, \theta_1)$, $(\rho_2, \theta_2)$ of the polar coordinates of $\mathbf{x}_1$ and $\mathbf{x}_2$ in $P$ is $((\theta_1-\theta_2)^2+(\log \rho_1/\rho_2)^2)^{\frac{1}{2}}$. The motions (distance-preserving transformations) of $R_n$ are of two types, the direct motions $\mathbf{x}'=M\mathbf{x}$ and the indirect motions $\mathbf{x}'=M\mathbf{x}/(\mathbf{x}, \mathbf{x})$ where $M$ is an "almost orthogonal" matrix, i.e., "orthogonal to within a constant." A "displacement" in $R_n$ is a motion for which some determination of the geodesic distance between a point and its transform is the same for all points. The displacements are characterized as direct motions with "skew matrices," a "skew matrix" being the sum of a skew-symmetric and a scalar matrix. A congruence of geodesics in $R_n$ is called a $C$-congruence if there exists a continuous one-parameter family $D_t$ of displacements which leave the congruence curvewise invariant. To every displacement $D:\mathbf{x}'=M\mathbf{x}$ there corresponds a $C$-congruence which is invariant under $D$, namely the integral congruence of the field (called $C$-field), $\mathbf{I}(\mathbf{x})=D(\mathbf{x})$. Any two curves $C_1$ and $C_2$ of a $C$-congruence are equidistant. The projections from $\mathbf{0}$ of $C_1$ and $C_2$ on a sphere with center $\mathbf{0}$ are parallel in the classic sense of Clifford. The only direct motions in $R_n$ which leave curvewise invariant a congruence of geodesics containing no circles are the displacements. The general form of a matrix $M$ expressing a displacement is given for $n$ even. For $n=4$, $M$ must be of one of the forms

$$M_L=\begin{Vmatrix} A & -B & -C & -D \\ B & A & D & -C \\ C & -D & A & B \\ D & C & -B & A \end{Vmatrix}, \quad M_R=\begin{Vmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{Vmatrix}.$$

In $R_n$, for given $\mathbf{x}_0$ and $\mathbf{I}_0$, there exist one matrix $M_L$ satisfying $\mathbf{I}_0=M_L\mathbf{x}_0$ and one matrix $M_R$ satisfying $\mathbf{I}_0=M_R\mathbf{x}_0$; thus $\mathbf{x}_0$ and $\mathbf{I}_0$ determine exactly two $C$-fields which may be described as left-hand and right-hand parallel. In the general case $C$-parallelism is defined by means of a class of $C$-fields such that any vector at any point belongs to exactly one of these fields. The existence of $C$-parallelisms in $R_8$ is shown by producing suitable classes of displacement matrices. For $n$ even, but arbitrary, a connection is defined which is nearly a $C$-parallelism.

*C. Y. Pauc.*

**Bol, Gerrit. Ein einfacher Beweis des Integralsatzes von Gauss-Bonnet.** Arch. Math. **2**, 199–201 (1950).

The classical Gauss-Bonnet theorem is proved as a consequence of the following theorem. Let $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$, $\mathbf{v}_3(t)$ be a unit, orthogonal frame of vectors which are differentiable and periodic functions of $t$ with period $w$. Let $O$ be the surface area inclosed on the unit sphere by the spherical image of $\mathbf{v}_1$. Then

$$\int_0^w \mathbf{v}_3\cdot d\mathbf{v}_2+O=2k\pi,$$

where $k$ is an integer. When $\mathbf{v}_1$ is the surface normal and $\mathbf{v}_2$ the tangent to a simple closed curve on the surface, this gives the Gauss-Bonnet theorem; for in this case $k=1$.

*C. B. Allendoerfer* (Haverford, Pa.).

**Allendoerfer, C. B. Characteristic cohomology classes in a Riemann manifold.** Ann. of Math. (2) **51**, 551–570 (1950).

This is a generalization of the Gauss-Bonnet formula to cover the Stiefel-Whitney characteristic cohomology classes, the Euler-Poincaré characteristic entering the Gauss-Bonnet formula being essentially the highest-dimensional Stiefel-Whitney class. However, with the exception of the highest-dimensional one, the Stiefel-Whitney classes are either of order two or to be taken mod 2. As a result the formulas are congruences and not equations, which is in the nature of the problem. The integrand is obtained by certain constructions in Euclidean space. Some results obtained during this process on the calculation by integrals of obstructions to extension are of independent interest. Once the integrand is constructed, the proof for a Riemann manifold is carried out as in the Gauss-Bonnet formula.

*S. Chern.*

**Gołąb, St. Généralisation des équations de Bonnet-Kowalewski dans l'espace à un nombre arbitraire de dimensions.** Ann. Soc. Polon. Math. **22** (1949), 97–156 (1950).

This paper is a study of the local structure of curves lying in a Riemannian $V_n$ and more particularly of those which lie in a $V_{n-1}$ which itself is in $V_n$. The general plan is to construct equations similar to the Frenet and Bonnet-Kowalewski equations and to interpret the various coefficients as curvatures of higher order.

Following Schouten and Struik [Einführung in die neueren Methoden der Differentialgeometrie, v. 2, 2d ed., Noordhoff, Groningen, 1938, p. 14] let $\mathfrak{i}_1$ be the unit tangent vector to the curve $C$, $\mathfrak{i}_k$ $(k=2, \cdots, n)$ its various normals relative to $V_n$, and $\kappa_k$ the corresponding curvatures. If $\kappa_{r-1}\neq 0$ and $\kappa_r=\cdots=\kappa_{n-1}=0$, the curve is called "flat of order $r$." Let $C\subset V_{n-1}\subset V_n$ and let $E_{n-1}$ be the tangent hyperplane to $V_{n-1}$ in $V_n$. Then $C$ is "asymptotic of order $p$" if $\mathfrak{i}_1, \cdots, \mathfrak{i}_p$ are in $E_{n-1}$ whereas $\mathfrak{i}_{p+1}$ is not, or if $C$ is asymptotic of order $p-1$ and $\kappa_p=0$. Let $\mathfrak{n}$ be the unit normal to $V_{n-1}$ in $V_n$. Then $C$ is called a "geodesic of order $p$" if $\mathfrak{n}$ is linearly dependent upon $\mathfrak{i}_1, \cdots, \mathfrak{i}_p$ and not upon $\mathfrak{i}_1, \cdots, \mathfrak{i}_{p-1}$, or if $C$ is a geodesic of order $p+1$ and $\kappa_{p-1}=0$.

It is shown that there exists an $n$-frame $\mathfrak{t}_1, \cdots, \mathfrak{t}_n$ such that $\mathfrak{t}_1=\mathfrak{i}_1$; $\mathfrak{t}_n=\mathfrak{i}_n$ and

$$\begin{aligned} D\mathfrak{t}_1 &= \textstyle\sum_\lambda \alpha_\lambda \mathfrak{t}_\lambda + \gamma \mathfrak{t}_n, \\ D\mathfrak{t}_\mu &= -\alpha_\mu \mathfrak{t}_1 + \beta_\mu \mathfrak{t}_n, \qquad (\lambda, \mu = 2, \cdots, n-1), \\ D\mathfrak{t}_n &= -\gamma \mathfrak{t}_1 - \textstyle\sum_\lambda \beta_\lambda \mathfrak{t}_\lambda, \end{aligned}$$

where $D$ indicates covariant differentiation. These are the generalized equations of Bonnet-Kowalewski. The curve $C$ is proved to be asymptotic of second order if $\gamma=0$, of the third order if $\gamma=0$, $\sum\alpha_\lambda\beta_\lambda=0$, of the fourth order if $\gamma=0$, $\sum\alpha_\lambda\beta_\lambda=0$, $\sum\alpha_\lambda'\beta_\lambda=0$. Conditions that $C$ be a geodesic of various orders are suggested. Relations are developed between the $\kappa_k$ and the $\alpha_\lambda$, $\beta_\lambda$, and $\gamma$.

An exposition of the author's $B$-curvatures for this situation is given [Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu.] **4**, 360–362 (1937)]. This results in equations related to those above. Connections between the "$m$-flatness" of the $B$-curvatures and the above "$r$-flatness" are developed. A number of theorems are proved relating the properties of being an $r$-flat curve, an asymptotic curve of order $p$, a geodesic of order $p$, a line of curvature, and a plane curve or straight line. The paper concludes with a detailed discussion of the case $n=3$.

*C. B. Allendoerfer* (Haverford, Pa.).

**Gołąb, St. Contribution à la théorie des objets géométriques.** Prace Mat.-Fiz. **47**, 1–15 (1949).

It is proved that in a one-dimensional space no special geometric objects exist of a class greater than 3. If only analytic functions are allowed, this theorem is equivalent to the well-known theorem proved by Lie that there do not exist transformations with a number of essential parameters greater than 3. But in this paper the proof is given for functions of class $C_1$ (partial derivatives of first order existing and continuous). *J. A. Schouten* (Epe).

**Gołąb, S. Sur les objets géométriques non différentiels.** Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1949**, 67–72 (1949).

The author considers special geometric objects of class zero [classification of Schouten and Haantjes, Proc. London Math. Soc. (2) **42**, 356–376 (1937)] with one component in a one-dimensional space. It is proved that all these objects that are not scalars are given by the formula

$$f(\Omega, \xi, \eta) = G\{g(\Omega, \xi), \eta\},$$

in which $G(u, v)$ is arbitrary and univalent with respect to $u$, $G(u, 0)$ is equal to $u$ and $g(u, v)$ is defined uniquely as the inverse of $G$ with respect to $u$. *J. A. Schouten* (Epe).

**Laptev, G. F. Differential connections of manifolds and their holonomy groups.** Doklady Akad. Nauk SSSR (N.S.) **71**, 597–600 (1950). (Russian)

The author states seven postulates on differentially connected spaces whose holonomy group is a finite Lie group. One starts with a "basic" topological space $B$ whose elements are "local" topological spaces $E$. The first two postulates define coordinate neighborhoods in each, $u^1, \cdots, u^N$ in $B$ and $x^1, \cdots, x^n$ in $E$. Postulates 3, 4, 5 may be combined into the definition of the mapping of $E(u+du)$ on $E(u)$ given by $x^i(u+du)\to x^i(u, du)=x^i(u)+f_J{}^i(x(u); u)du^J+pt^i$ or equivalently as (*) $dx^i=f_J{}^i(x(u); u)du^J$. Postulate 6 is divided into: (a) $B$ contains admissible lines each being continuous and consisting of a finite number of piecewise analytic arcs; (b) a finite sum of admissible lines is admissible; (c) every point $M(u)$ may be connected to a fixed point $M(u_0)$ by an analytic admissible line; (d) through each point in each direction there is an admissible line. The mappings of $E(u)$ on itself defined by the solutions of (*) corresponding to all closed curves through $M(u)$ define the holonomy group of the space, this group being the same for all points. The last postulate requires this group to be an $r$-parameter Lie group. For this to be the case (*) must have the form $dx^i=\xi_p{}^i(x)\omega^p$ $(p=1, \cdots, r)$, where $\omega^p=\Gamma_J{}^p(x(u))du^J$. Thus the classification of connections of $N$-dimensional manifolds of $n$-dimensional local spaces having a finite holonomy group is equivalent to the classification of Lie groups having an $n$-dimensional representation. The author gives the four possible types of connection for $n=1$.

*M. S. Knebelman* (Pullman, Wash.).

**Denk, Franz, und Haupt, Otto. Über die Windungsmonotonie der Elementarbogen im $\mathfrak{R}^{(n)}$.** J. Reine Angew. Math. **187**, 95–108 (1949).

An arc $\mathfrak{B}$ of order $n$ in affine $n$-space meets any (linear) $(n-1)$-space in not more than $n$ points. If $k+1$ different points of $\mathfrak{B}$ converge to an end-point $O$ of $\mathfrak{B}$, then the $k$-space through them has a unique limit space $\mathfrak{T}^{(k)}$, the osculating $k$-space of $\mathfrak{B}$ at $O$. The $k$-half-space bounded by $\mathfrak{T}^{(k-1)}$ and going through a point of $\mathfrak{B}$ different from $O$ converges to a limit half-space $\mathfrak{Th}^{(k)}\subset\mathfrak{T}^{(k)}$ if that point approaches $O$. By means of the $\mathfrak{Th}^{(k)}$'s an osculating coordinate system can be constructed. Its $k$th axis is an oriented straight line leading from $O$ into $\mathfrak{Th}^{(k)}$. Let $\mathfrak{Q}$ be the $(n-1)$-space through a set of $n$ different interior points of $\mathfrak{B}$ which lie sufficiently close to $O$, and let $c_k$ be the $k$th coordinate of the point at which $\mathfrak{Q}$ intersects the $k$th axis. The authors prove that, if the points of the set converge to $O$ monotonically, then the numbers $c_k/c_{k-1}$ converge to 0 monotonically from below and the intersection of $\mathfrak{Q}$ with $\mathfrak{Th}^{(k)}$ converges to $\mathfrak{Th}^{(k-1)}$, also monotonically in a certain sense $(k=1, 2, \cdots, n;$ $c_0=-1;$ $\mathfrak{Th}^{(0)}=O)$.

Reviewer's remark. If $\mathfrak{B}$ is differentiable, i.e., has osculating spaces of all dimensions at each point, then we may choose for $\mathfrak{Q}$ the osculating $(n-1)$-space of a point different from $O$. Thus the last result contains (essentially) a generalization of the theorem that the dual curve of a differentiable curve $\mathfrak{B}$ is again differentiable. *P. Scherk.*

*Delachet, André. **Calcul vectoriel et calcul tensoriel.** Presses Universitaires de France, Paris, 1950. 128 pp.

This small volume, no. 418 of the series "Que sais-je?," is not offered by the author as a complete account of vector and tensor calculus, nor as a mathematician's book. His object is to help science students in the understanding of standard works on geometry and physics. The book consists of three parts. Part I (vector algebra) begins in an elementary way with the theory of free vectors in 3 dimensions, a vector being initially defined as a segment of line. Scalar and vector products are defined and the usual formulae obtained, with an indication of their application to trigonometry and geometry. The second chapter deals with line vectors (vecteurs glissants) and moments. Part II (vector analysis), still 3-dimensional, has two chapters of which the former is devoted to the differentiation and integration of vector functions of one or more parameters, with application to the geometry of twisted curves and of surfaces, and the latter to the theory of vector fields (grad, div, curl, Green, Gauss, Stokes, etc.). Part III (tensor algebra) has four chapters dealing respectively with (i) the $n$-dimensional affine space $E_n$ of vectors, (ii) covariance, contravariance and the dual space $E_n^*$, (iii) tensors and tensor algebra, (iv) inner product and the consequent metric geometry. Part IV (tensor analysis) contains three chapters on (i) tensor fields in affine space, and their differentiation, (ii) tensors and tensor fields in any $n$-dimensional manifold, with a proof that the curl of a vector is a tensor, but without any theory of covariant differentiation, (iv) a 3-page introduction to the idea of Riemannian space. *H. S. Ruse.*

**Nikol'skiĭ, K. V. Killing's equation and the fundamental metric tensor.** Doklady Akad. Nauk SSSR (N.S.) **72**, 277–279 (1950). (Russian)

The author raises the question of whether a field theory could be based on a third order tensor $c^{\gamma}{}_{\alpha\beta} = -c^{\gamma}{}_{\beta\alpha}$, in terms of which the ordinary metric could be defined as $g_{\alpha\beta} = c^{\lambda}{}_{\alpha\mu}c^{\mu}{}_{\beta\lambda}$. He obtains equations satisfied by $c^{\gamma}{}_{\alpha\beta}$ which are analogous to those of Killing for $g_{\alpha\alpha}$. *A. J. Coleman.*

## NUMERICAL AND GRAPHICAL METHODS

**Van Wijngaarden, A., and Scheen, W. L. Table of Fresnel integrals.** Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **19**, no. 4, 26 pp. (1949).

This paper consists almost entirely of an excellent 20-page table of the integrals

$$C(u) = \int_0^u \cos \tfrac{1}{2}\pi t^2 dt \quad \text{and} \quad S(u) = \int_0^u \sin \tfrac{1}{2}\pi t^2 dt.$$

Five decimal values of both functions are given for $u = 0(0.01)20$, with modified second differences throughout to assist interpolation. The range includes 200 turning values of each function. A misprint has been marked by the authors in the copy received for review: $C(4.95)$ should read 54504. An introduction describes the calculations and gives 16 or more decimal coefficients in expansions of $C(u)$ and $S(u)$ in powers of $u$. *J. C. P. Miller* (London).

**Kellogg, Myron G. An extension of Fresnel integral tables.** NAVORD Rep. **1064** (NOTS 184), U. S. Naval Ordnance Test Station, Inyokern, Calif., iii+5 pp. (1949).

This gives values of

$$C(x) = \int_0^x \cos \tfrac{1}{2}\pi x^2 dx, \qquad S(x) = \int_0^x \sin \tfrac{1}{2}\pi x^2 dx$$

to six decimals, with second differences, for $x = 6(.02)7$. The method of computation is described and should result in accuracy within less than a unit in the sixth decimal. The second differences given are sufficient, by themselves, only for interpolation to 4 decimals. For another table of these integrals see the preceding review. *J. C. P. Miller.*

**Huggins, W. H. Tables of Bessel function derivatives** Air Materiel Command, Cambridge Field Station, Cambridge, Mass., Rep. no. 4-3, i+3 pp. (1947).

The tables in this report give values to 4 decimals (or 4 significant figures) of the Bessel function derivatives $J_n'(x)$, $Y_n'(x)$ for $n = 1, 2, 3$ and $x = 0(0.2)9.8$. It is stated of the values that "it is believed that they may be relied upon to three figures." A check of about 100 of the 300 values indicates that this claim is justified, errors up to 4 units in the fourth decimal having been found. *J. C. P. Miller.*

**Horton, C. W. A short table of Struve functions and of some integrals involving Bessel and Struve functions.** J. Math. Physics **29**, 56–58 (1950).

This paper gives, for $x = 0(.1)10$, values of $H_n(x)$, $n = 2, 3, 4$, $C_n(x)$, $n = 1, 2, 3^*, 4^*$, $D_n(x)$, $n = 0, 1, 2, 3^*, 4^*$ to 4 decimals (3 decimals for $n > 6$ in cases indicated by an asterisk). In this $H_n(x)$ is a Struve function and $C_n(x) = \int_0^x z^n J_n(z)dz$, $D_n(x) = \int_0^x z^n H_n(z)dz$. *J. C. P. Miller* (London).

**Abramowitz, Milton. Tables of integrals of Struve functions.** J. Math. Physics **29**, 49–51 (1950).

Values of $\overline{H_0(x)}$, $\overline{H_1(x)}$, $\overline{L_0(x)}$, $\overline{L_1(x)}$ are given to 6 decimals when $x \leqq 5$, and 6 significant figures elsewhere for $x = 0(.1)10$, where the notation $\overline{y(x)} = \int_0^x y(t)dt$ is used, and $H_\nu(x)$, $L_\nu(x)$ are the Struve and modified Struve functions respectively. *J. C. P. Miller* (London).

*Tables of the Generalized Exponential-Integral Functions, by the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. XXI. Harvard University Press, Cambridge, Mass., 1949. xxvi+416 pp. $8.00.

This volume is a companion to two earlier volumes in the same series [Tables of Generalized Sine- and Cosine-Integral Functions. Parts I and II. Harvard University Press, Cambridge, Mass.; these Rev. **10**, 572]. It gives values of the three functions:

$$\mathrm{E}(a, x) = \int_0^x \frac{1 - \exp\{-(a^2+x^2)^{\frac{1}{2}}\}}{(a^2+x^2)^{\frac{1}{2}}} dx$$

$$\mathrm{Es}(a, x) = \int_0^x \frac{\exp\{-(a^2+x^2)^{\frac{1}{2}}\} \cdot \sin(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}} dx$$

$$\mathrm{Ec}(a, x) = \int_0^x \frac{1 - \exp\{-(a^2+x^2)^{\frac{1}{2}}\} \cdot \cos(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}} dx.$$

Six decimals are given, together with first differences both $x$-wise and $a$-wise. Argument ranges for both $x$ and $a$ are as follows, for the various tables: I. 0(.001).049; II. 0(.002).098; III. 0(.005).245; IV. 0(.01).99; V. 0(.02)1.98; VI. 0(.05)2.45; VII. 0(.1)4.9; VIII. 0(.2)9.8. The arguments $x$ and $a$ vary

independently through the ranges indicated. $\Delta x$ and $\Delta a$ suffice for interpolation in tables I and II, but second differences are needed for full accuracy in the other tables. The arrangement is better than for the earlier volumes mentioned above and is satisfactory for both $a$-wise and $x$-wise interpolation. The introduction outlines properties of the functions, methods of computation and of interpolation, etc. Some relations seem worth quoting: Since the Bessel function $K_0(a)$ is given by

$$K_0(a) = \int_0^\infty \frac{\exp\{-a(1+t^2)^{\frac{1}{2}}\}}{(1+t^2)^{\frac{1}{2}}}dt,$$

the function

$$\bar{\mathrm{E}}(a, x) = \sinh^{-1}\frac{x}{a} - \mathrm{E}(a, x) = \int_0^x \frac{\exp\{-(a^2+x^2)^{\frac{1}{2}}\}}{(a^2+x^2)^{\frac{1}{2}}}dx$$

is an incomplete Bessel function. Note also that

$$\mathrm{E}(0, x) + \mathrm{Ei}(-x) = \ln x + \gamma$$

is the special case $a=0$. *J. C. P. Miller* (London).

*Tables of the Function $\frac{\sin\phi}{\phi}$ and of its First Eleven Derivatives, by the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. XXII. Harvard University Press, Cambridge, Mass., 1949. xviii+241 pp. $8.00.

As indicated in its title, this table gives the values of the sine-integral ordinate $g(\phi)=\sin\phi/\phi$, and of its first eleven derivatives $(d^r/d\phi^r)g(\phi)$ for $(180/\pi)\phi=0(\frac{1}{2})3599\frac{1}{2}$; in other words, argument is given in degrees. Values are to 9 decimals, usually about 8 significant figures, and accuracy to 0.7 of a unit in the final digit is claimed. The twelve function values for each $\phi$ are given across an opening, with 60 lines to a page, giving a good, useful, well-arranged and compact table. *J. C. P. Miller* (London).

**MacDonald, A. D. Tables of the function $e^{-az/\gamma}M(\alpha; \gamma; z)$.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. no. 130, ii+10 pp. (1949).

This report gives 6-figure or 6-decimal values of the function $M(z)=e^{-az/\gamma}M(a, \gamma; x)$ in which

$$M(\alpha, \gamma; x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

is the confluent hypergeometric function; values are given for $\gamma=\frac{1}{2}(\frac{1}{2})2$, $\alpha=.001, .01, .05, .1(.1)1$, also .25 and .75 with $z=0.1(.1)1(.5)8$. They were computed by the Joint Computing Group of the Massachusetts Institute of Technology. *J. C. P. Miller* (London).

*Tables of the Binomial Probability Distribution. National Bureau of Standards, Applied Mathematics Series, No. 6. United States Government Printing Office, Washington, D. C., 1950. x+387 pp. $2.50.

Table I gives individual terms $\binom{n}{r}p^r q^{n-r}$ in the expansion of the binomial $(q+p)^n$, and table II gives the partial sums $\sum_{r=r'}^{n} p^r q^{n-r}$. In each case seven decimal values are given for $p=1-q=.01(.01).50$, $n=2(1)49$, $r=0(1)n-1$. A short introduction describes the preparation of the tables, their interpolation, and lists several other similar tables, mostly unpublished, which indicates the need for tables such as these, and an alarming amount of overlapping computational effort. This, the first real book in this series, is bound in cloth with stiff covers. *J. C. P. Miller* (London).

**Greville, Thomas N. E. On the derivation of discrete interpolation formulas.** Soc. Actuar. Trans. 1, 343–357; discussion, 358–368 (1949).

Consider the formulae

$$v_n = \sum_{k=-\infty}^{\infty} L_{n-km}u_{km}, \quad V_n = m^{-1}\sum_{r=-\infty}^{\infty} L_{n-r}U_r,$$

where the $L_r$ are, say, Lagrangian interpolation coefficients, and $m$ is a given positive integer. The first formula subdivides the interval of tabular values $u_{km}$ into $m$ aliquot parts, the second smooths the data $U_t$. If we write $U_t=mu_{km}$ when $t=km$, and $U_t=0$ otherwise, we have $V_n=v_n$. The author uses this principle to deduce certain known actuarial discrete interpolation formulae. *L. M. Milne-Thomson.*

**Guest, P. G. Orthogonal polynomials in the least squares fitting of observations.** Philos. Mag. (7) 41, 124–137 (1950).

The author's summary is as follows. The theory of least-squares curve-fitting of nonequidistant data is developed in terms of orthogonal polynomials and power moments. A modification of the Doolittle scheme is proposed which enables probable error calculations to be carried out very simply. The modified scheme is illustrated by a specific example. *T. N. E. Greville* (Washington, D. C.).

**Jenne, W. Zur Auflösung linearer Gleichungssysteme.** Astr. Nachr. 278, 73–95 (1949).

Least-squares equations for adjustment of triangulations are solved following a procedure of the geodeticist K. Friedrich; a principal tool is the inversion of Jacobi matrices by continued fractions. Certain related polynomials of N. E. Nörlund are studied. *A. T. Lonseth.*

**Wolf, H. Das Verfahren der schrittweisen Annaeherung in der Ausgleichung von Nivellementsnetzen.** Bull. Géodésique N.S. 1950, no. 15, 52–62 (1950).

**Collatz, L. Iterationsverfahren für komplexe Nullstellen algebraischer Gleichungen.** Z. Angew. Math. Mech. 30, 97–101 (1950). (German. English, French and Russian summaries)

A method for the computation of the roots of an algebraic equation with real or complex coefficients is described. The method of iteration consists of the multiplication of the given equation by $z^r$ so that it may be written in the form $(z^k-q)^p=R(z)$, where $R(z)$ is a known rational function of $z$. Then, for an approximation $z_0$, a sequence $z_\nu$, $\nu=0, 1, \cdots$, is defined by the relation $(z_{\nu+1}^k-q)^p=R(z_\nu)$. The sequence $z_\nu$ is shown to converge to a root of the given equation for the appropriate choice of $k$ and $p$ and the real number $r$, when the initial value $z_0$ is chosen sufficiently close to the root. Examples are given in which this method gives approximate values of the complex roots more rapidly than Newton's method and also gives several complex roots simultaneously. *E. Frank* (Chicago, Ill.).

**Basile, R. Résolution de systèmes d'équations linéaires algébriques et inversions de matrices au moyen des machines de mécanographie comptable. Complément pratique par R. Janin.** Office National d'Études et de Recherches Aéronautiques, Paris, publ. no. 28, v+21 pp. (4 plates) (1949).

This is a program, complete with plug-board diagrams and estimates of time and cost, for carrying out the solution of a set of $n$ linear algebraic equations in $n$ unknowns.

The equipment consists of a multiplier, a reproducer (with a rather large number of selectors), and a collator, all made by the "firme BULL." The mathematical method is the standard procedure for diagonalizing a matrix. There are $n$ stages with a separate set of cards for each stage. In the $i$th stage one starts with a set of cards containing $A_{pq}$ where $p$ runs from 1 to $n$ and $q$ from $i$ to $n$; by two runs through the reproducers it is arranged that $A_{pq}$, $A_{(q)}$, $A_{pi}$ and $A_{ii}$ all appear on the same card; then a run through the multiplier calculates the second order determinants so as to prepare for the next stage. *H. B. Curry* (Louvain).

**Polachek, H. On the solution of systems of linear equations of high order.** Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLM-9522**, 8 pp. (1948).

This is a preliminary report on the accuracy of an iterative scheme for solving linear algebraic systems by digital machines. If $x^{(1)}$ is an approximate solution of $Ax=y$ ($y$ is a known $n$-vector, $A$ is a nonsingular $n\times n$ matrix, $n$ large), it is proposed to construct successive approximations $x^{(2)}, x^{(3)}, \cdots$ to $x$ such that $x^{(i+1)}=x^{(i)}+v^{(i)}$, where $v^{(i)}=A^{-1}(y-Ax^{(i)})$. Accumulation of round-off error is analyzed by the methods of von Neumann and Goldstine [Bull. Amer. Math. Soc. **53**, 1021–1099 (1947); these Rev. **9**, 471], it being assumed that the inverse $A^{-1}$ of $A$ has been approximated as they suggest. The error-estimates indicate that a certain "residual error" cannot be entirely eliminated by repeated application of the process. The labor involved in each iteration-cycle is shown to be $O(n^2)$, as compared with $O(n^3)$ for the matrix-inversion. *A. T. Lonseth.*

**Andersen, Einar. Solution of great systems of normal equations.** Bull. Géodésique N.S. **1950**, no. 15, 19–29 (1950).

Cf. Mém. Inst. Géodésique Danemark [Geodætisk Instituts Skr.] (3) **11** (1947); these Rev. **9**, 622.

**Dreyer, H. J. Solution of systems of linear equations by means of punched-card machines.** Headquarters Air Materiel Command, Wright Field, Dayton, Ohio, Rep. no. **F-TS-1046-RE**. iii+13 pp. (1946).

A verbal account is given of the solution of a system of linear equations by means of punch card equipment. The method is an elimination procedure in which the appropriate multipliers are chosen nonautomatically. The equipment used is a multiplier, tabulator, reproducer, and a sorter. The results of the method when applied to a system of the 20th order are illustrated. The work was carried to 7 significant figures, the answers being good to only two, after a corrective rerun. The time required was 65 hours. This report was originally dated 1943. *D. H. Lehmer* (Berkeley, Calif.).

**Sherman, Jack, and Morrison, Winifred J. Adjustment of an inverse matrix corresponding to a change in one element of a given matrix.** Ann. Math. Statistics **21**, 124–127 (1950).

A method is presented for determining the new inverse matrix corresponding to a change in one element of a given matrix. The method requires only a simple computation to determine each element of the new matrix from the original inverse matrix. A proof of the validity of the method is given. In an illustrative numerical example, the step-by-step computation is clearly shown. A discussion is given of the case where a singular matrix results from changing one element of the given matrix. The authors point out that, if two or more elements in the original matrix are to be changed, the new inverse can be found by successive applications of the method. *S. Levy* (Washington, D. C.).

**Booth, A. D. An application of the method of steepest descents to the solution of systems of non-linear simultaneous equations.** Quart. J. Mech. Appl. Math. **2**, 460–468 (1949).

The author studies the convergence of several iterative methods for solving a set of nonlinear simultaneous equations. A "tangent descent" method of solution is presented which is particularly suitable in those cases where it is desired to find minima. The author considers the Southwell relaxation procedure best for a hand computing machine and either the "tangent" or the "steepest" descent methods best for an electronic computer. *S. Levy.*

**Beck, Eugen. Zwei Anwendungen der Obreschkoffschen Formel.** Z. Angew. Math. Mech. **30**, 84–93 (1950). (German. English, French and Russian summaries)

Obreschkoff's formula connects the quantities

$$y(x), y'(x), \cdots, y^{(k)}(x), y(x_0), y'(x_0), \cdots, y^{(m)}(x_0)$$

in a linear relationship with coefficients depending only on $k$, $m$, and $h=x-x_0$. The author uses this formula together with the differential equation and the successive equations obtained by differentiation of the differential equation to eliminate the derivatives of $y(x)$ and thus secure a formula for $y(x)$ in terms of $y(x_0), y'(x_0), \cdots, y^{(m)}(x_0)$. Actually an explicit solution is obtainable only for linear equations. For nonlinear equations he resorts to successive approximations. The method is extended to systems of equations and equations of higher order, and is illustrated by a number of examples. [In contrast to a similar method given by the reviewer [J. Research Nat. Bur. Standards **43**, 537–542 (1949); these Rev. **11**, 619] which is entirely numerical, Beck obtains approximate analytical expressions for the solution.] *W. E. Milne* (Corvallis, Ore.).

**Hudson, G. E. On the explicit numerical solution of linear homogeneous differential equations with constant coefficients.** J. Math. Physics **29**, 52–55 (1950).

A method of calculating solutions satisfying given initial conditions for the case of repeated roots. *P. Franklin.*

**McDonald, Donald. Analog computers for servo problems.** Rev. Sci. Instruments **21**, 154–157 (1950).

The impedance equation connecting input and output voltage for an amplifier is specialized to correspond to (1) differentiation, (2) integration and (3) summation. In the first two cases distortion in phase and magnitude occurs, both depending on frequency. Under certain assumptions satisfied in practice, limits are found for these distortion terms in the case of an amplifier of gain 700, the design details of which are given. It is concluded that such low gain amplifiers are suitable for servo problem analogue computers. *R. Church* (Annapolis, Md.).

**Macnee, Alan B. A high-speed product integrator.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. no. **136**, i+21 pp. (1949).

This is a technical description of an electronic analogue machine for the evaluation of integrals of the form

$$F(y)=\int_a^{b \text{ or } s} f(x)K(x, y)dx.$$

There are two inputs (for $x$ and $y$); the solution is displayed

on a cathode ray screen. A one-hundred point solution for $F(y)$ is obtained every 1.67 sec. The principal parts of the machine are a function generator for $f(x)$, a kernel generator for $K(x, y)$, a multiplier, and an integrator. The first, third, and fourth of these components are taken directly from the electronic differential analyzer [same Tech. Rep. no. 90 (1948); these Rev. 11, 465]. The function generator uses photoelectric tracking of a mask on the face of a cathode ray tube; a method of making such a mask which is continuously adjustable is being developed. The kernel generator depends on the problem. Since the $y$ has to change only slowly, it can be generated by motor-driven potentiometers. Methods of generating $K(x, y)$ in this way in combination with components of the differential analyzer are given for three special cases, viz.: Fourier transforms, solutions of the Schlömilch integral equation, and the superposition (or convolution) integral. Possibilities for more general sorts of kernel generation are discussed briefly. An error analysis is made and the results of experimental tests on the three principal cases are given. The errors depend on the problem; for most cases the observed error is around 2%, but had a maximum of 10% in the worst case.

*H. B. Curry* (Louvain).

**Smirnov, S. V. On the problem of general anamorphosis.** Doklady Akad. Nauk SSSR (N.S.) **69**, 297–300 (1949). (Russian)

The equation $F(x, y, z)=0$ is equivalent to the vanishing of a determinant of Massau if and only if a system of two partial differential equations given by T. H. Groenwall [J. Math. Pures Appl. (6) **8**, 59–102 (1912), p. 70] have a common solution. In this paper an equivalent set of three equations is obtained by introducing two new dependent variables $u$ and $v$ by means of the relations

$$2\partial C/\partial y+\partial D/\partial x=3e^u, \quad \partial C/\partial y+2\partial D/\partial x=3e^v,$$

it being understood that we do not have the well-known special case $\partial^2 \ln M/\partial x\partial y=0$. Successive differentiation of an equation of the new system, and elimination, shows that it is possible to express each of the first partial derivatives of $u$ and $v$ as an algebraic function of $e^u$, $e^v$ and the partial derivatives of $M$ up to those of sixth order. Further consideration of these derived equations leads to the following conclusion: necessary and sufficient conditions for the existence of an anamorphosis of $F(x, y, z)=0$ can be expressed by the vanishing (or nonvanishing) of polynomials with coefficients which are rational functions of $M$ and its partial derivatives; these conditions can be obtained by a finite number of differentiations and eliminations.

*R. Church* (Annapolis, Md.).

**Vil′ner, I. A. Reduction of a nomographable analytic relation to normal form.** Doklady Akad. Nauk SSSR (N.S.) **69**, 3–6 (1949). (Russian)

The author previously gave [same Doklady (N.S.) **63**, 99–102 (1948); these Rev. **10**, 577] two mutually inverse canonical forms to which the analytic relation $F(x, w)=0$ is reducible if it can be nomographed with not all scales straight (i.e., nonzero genus). Here he gives practical details for effecting this reduction and formulas for certain of the constants not explicitly determined in the previous work. When the discriminant and roots of the Weierstrass elliptic function, which occurs under the radical of the canonical form, and the additive constant satisfy appropriate special conditions there result various nomographic relations of the first class (i.e., those with no more than two of the four scales curved) previously discussed [C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 783–786 (1947); these Rev. **9**, 106; and elsewhere]. In general, if $F(z, w)=0$ is nomographable, both $z$ and $w$ can be expressed as Legendre's elliptic integral of the first kind with an auxiliary complex variable $\sigma$ as argument so that a nomogram is possible with two straight scales for $\sigma$ as a parameter in addition to the four scales for $z$ and $w$.

*R. Church* (Annapolis, Md.).

## ASTRONOMY

**Koziel, K. The differential formulae of spherical polygonometry and their applications to astronomy.** Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1949**, 1–16 (1949).

Some problems in spherical astronomy require the introduction of differential corrections to the elements of spherical polygons. These problems have frequently been solved by dividing them into several stages in which the differential corrections to successive spherical triangles are introduced. Owing to the introduction and subsequent elimination of auxiliary quantities this procedure may become complicated. The practical difficulties may be overcome by the use of matrix methods. For applications especially to spherical astronomy Banachiewicz introduced a modified type of matrix, called cracovians. For convenience in numerical applications the multiplication rule with cracovians is changed to column by column multiplication. This article presents a general theory of differential corrections to spherical polygons with the use of cracovians. Detailed examples are given to illustrate the method. *D. Brouwer*

**Il′inskiĭ, I. I. On the great circle case in the calculation of orbits.** Akad. Nauk SSSR. Astr. Zhurnal **27**, 57–60 (1950). (Russian)

The author completes his earlier work on the determination of orbits [Astr. J. Soviet Union [Astr. Zhurnal] **23**, 367–376 (1946); these Rev. **8**, 409] by showing how to modify his methods of that paper to handle the singular case arising when the three observations lie on an arc of a great circle.

*R. G. Langebartel* (Urbana, Ill.).

**Taylor, Stewart W. On periodic orbits in the restricted problem of three bodies.** Astr. J. **55**, 33–38 (1950).

Polar coordinates in a uniformly rotating coordinate system are used for the analysis of periodic orbits. The formulae required for computations based on first-order perturbations are developed and numerical applications are given for periodic orbits near the commensurabilities 3/2, 2/1, 3/1 and 7/4. Special reference is made to earlier work along similar lines by L. Hopkins [Astr. J. **29**, 81–97 (1916); **36**, 25–33 (1924)].

*D. Brouwer* (New Haven, Conn.).

**Buchdahl, H. A. On Tolman's equation describing the thermal equilibrium in a gravitating sphere of fluid.** Philos. Mag. (7) **41**, 362–363 (1950).

Alternative, and somewhat improved, derivation of Tolman's relation $T\sqrt{g_{44}}=$constant for the temperature of a fluid sphere in thermal equilibrium.

*H. P. Robertson.*

**Scherrer, W. Über die Gravitation kontinuierlich ausgebreiteter Massen.** Comment. Math. Helv. **24**, 46–63 (1950).

The hydrodynamical equations governing a spherical distribution of mass in equilibrium under its own gravitation are

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{4\pi\kappa}{r^2}\int_0^r \mu r^2 dr,$$

$$\frac{\partial \mu}{\partial t}+\frac{\partial}{\partial r}(\mu v)=-\frac{2\mu v}{r},$$

where $v=v(r,t)$ is the velocity in the direction of the radius $r$, $\mu=\mu(r,t)$ is the density and $\kappa$ is the constant of gravitation. The author shows that these equations admit solutions of the form $v(r,t)=r\omega(t)$ and $\mu=\mu(t)$, where $\omega$ and $\mu$ are functions only of the time $t$; for with these substitutions the hydrodynamical equations become $\dot{\omega}=(2/3\mu)\omega^2+(8\pi\kappa/9)$ and $\dot{\mu}=-3\omega\mu$, where dots denote differentiation with respect to the time. The equations for $\omega$ and $\mu$ can be readily integrated to give

$$\omega=(8\pi\kappa/3)^{\frac{1}{2}}\mu^{\frac{1}{2}}(\mu^{\frac{1}{2}}-A^{\frac{1}{2}})^{\frac{1}{2}},$$

$$t=\frac{3}{(24\pi\kappa)^{\frac{1}{2}}}\int\frac{d(\mu^{-\frac{1}{2}})}{(\mu^{\frac{1}{2}}-A^{\frac{1}{2}})^{\frac{1}{2}}},$$

where $A$ is a constant of integration. The cases (i) $A>0$, (ii) $A=0$ and (iii) $A<0$ should be distinguished. In case (i) the density first decreases and then increases to infinity as the whole system collapses to a point in a finite time; in case (ii) the density and the velocity monotonically tend to zero as $t\to\infty$; and finally in case (iii) the density tends to zero but the velocity tends to a finite limit as $t\to\infty$. [The system considered here is related to the system of Newtonian cosmology discovered by Milne and McCrea.]

*S. Chandrasekhar* (Williams Bay, Wis.).

**Van Wijk, Uco. On the dynamics of galactic clusters.** Ann. Astrophysique **12**, 81–95 (1949).

In this paper an attempt is made to generalize an earlier investigation of Mineur [Ann. Astrophysique **2**, 1–244, 328 (1939)] on the dynamics of galactic clusters to the case when the cluster is not necessarily in equilibrium. [It appears to the reviewer that the fundamental premise of the paper is erroneous. The virial theorem giving the rate of change of the moment of inertia of a system of mass points under their mutual attraction and in an external gravitational field having an axial symmetry is well known [cf. S. Chandrasekhar, Principles of Stellar Dynamics, University of Chicago Press, 1942, pp. 215–219; these Rev. **4**, 57]. The reviewer sees no justification for setting this term equal to zero for a cluster not in a stationary state; but this is what the author does.] *S. Chandrasekhar.*

**Bondi, C. M., and Bondi, H. The integration of the equations of stellar structure.** Monthly Not. Roy. Astr. Soc. **109**, 62–85 (1949).

It is well known that the equations governing the equilibrium of a star admit certain homology transformations and that in fitting two regions in which different opacity laws or different laws of the mode of heat transfer (convection, conduction) are valid, it is convenient to carry out the solution in two stages. First, the fitting is considered for the homology invariant combinations of the physical variables. Once this has been accomplished the solution can be completed in terms of the eigenvalues of certain parameters which occur in the equations of equilibrium when expressed in nondimensionless form. The variables which have been in general use are $U=4\pi r^3\rho/M(r)$ and $V=GM(r)\rho/Pr$, where $\rho$ is the density, $P$ is the pressure and $M(r)$ is the mass interior to $r$. It is known that a first order equation governs $U$ and $V$; this enables the elimination of the constants of homology in the fitting process. [Cf. S. Chandrasekhar, An Introduction to the Study of Stellar Structure, University of Chicago Press, 1939, p. 352 and elsewhere.] The present paper suggests that the use of the variables $S=U/V$ and $Q=1/V$ may be more convenient for certain purposes.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Severnyĭ, A. B. On the stability and oscillations of gaseous spheres and stars.** Akad. Nauk SSSR. Izvestiya Krymskoĭ Astrofiz. Observ. **1**, no. 2, 3–63 (1948). (Russian)

The author sets out to study the vibrational stability of the stars by the method of small oscillations. The dependence of the stability on the ratio $\gamma$ of specific heats is considered; and the author reiterates the well-known result that, in the absence of dissipative forces, the stars are unstable for $\gamma<4/3$. Turbulence and radiative viscosity are shown to tend to stabilize the star. The dissipation of energy in stars of moderate masses is found to be much larger than the excess of heat produced by nuclear reactions. On the basis of these considerations, the author constructed a stability diagram for various configurations of gaseous stars. Stars of small masses (particularly the white dwarfs) are found to be always stable. For large masses, two groups of stars may be distinguished, depending on whether $\gamma<4/3$ or $\gamma>4/3$. The absence of stars from unstable regions is invoked to explain certain well-known features of the Hertzsprung-Russell diagram.

A relation between the period and mean density of pulsating gas spheres is derived, claiming a better agreement with the observations than previously reached by Eddington or Ledoux and Pekeris. The turbulent damping of pulsations is found to be roughly proportional to the mean density; consequently, the stars with short periods should, on the average, exhibit smaller amplitudes. A slow rotation is shown to make a star unstable with respect to small non-radial oscillations. *Z. Kopal* (Cambridge, Mass.).

**Prasad, Chandrika. Radial oscillations of a particular stellar model.** Monthly Not. Roy. Astr. Soc. **108**, 414–416 (1948).

The equation for radial oscillations of a star is solved for the case of the homogeneous star with a point mass at the centre. An algebraic solution is possible when the point mass is one third of the total mass of the star. This solution is the only algebraic solution of the oscillation equation apart from the original polytropic models which were studied by Eddington and later by Sterne. On comparing the models of the present calculation with the cases $\delta$ Cephei and X Cygni and RR Lyrae it is found that the methods used give the right order of periods for cepheid as well as cluster-type variables. *G. Randers* (Oslo).

**Prasad, Chandrika. Radial oscillations of a stellar model.** Monthly Not. Roy. Astr. Soc. **109**, 103–107 (1949).

In the present paper an investigation similar to that reviewed above is made for a star with the density varying as $(1-r^2)/R^2$. In this case also the equation can be solved

without numerical integration. It is shown that the periods of models with a continuous density distribution vary according to the density conditions. The model calculated in this paper gives a period of 6.1 days for $\delta$ Cephei. This period, however, depends on the choice of the ratio of specific heats. *G. Randers* (Oslo).

## MECHANICS

### *Hydrodynamics, Aerodynamics, Acoustics*

**Prim, R., and Truesdell, C. A derivation of Zorawski's criterion for permanent vector-lines.** Proc. Amer. Math. Soc. 1, 32–34 (1950).

The authors set out three purely kinematical questions: (1) What is a necessary and sufficient condition that the strength of the vector-tubes of a continuously differentiable vector field $\mathbf{c}$ be the same at all cross-sections? (2) Given a continuously differentiable velocity field $\mathbf{v}(\mathbf{r}, t)$, what is a necessary and sufficient condition that the strength of the vector-tubes of a second continuously differentiable field $\mathbf{c}(\mathbf{r}, t)$ remain constant throughout the motion? (3) Given a continuously differentiable velocity field $\mathbf{v}(\mathbf{r}, t)$, what is a necessary and sufficient condition that the vector-tubes of a second continuously differentiable field $\mathbf{c}(\mathbf{r}, t)$ be material tubes?

The authors state, with full references to original and alternative derivations, the answers to these questions given (1869–1900) by Kelvin, Lamb and Zorawski, respectively

$$\text{(1)} \qquad \operatorname{div} \mathbf{c} = 0,$$

$$\text{(2)} \qquad D\mathbf{c}/Dt - \mathbf{c} \cdot \operatorname{grad} \mathbf{v} + \mathbf{c} \operatorname{div} \mathbf{v} = 0,$$

$$\text{(3)} \qquad \mathbf{c} \times [D\mathbf{c}/Dt - \mathbf{c} \cdot \operatorname{grad} \mathbf{v} + \mathbf{c} \operatorname{div} \mathbf{v}] = 0.$$

They supply, as the main purpose of the paper, a simple vectorial derivation of (3). They then remark: "Since (3) is a consequence of (2), but (2) is not generally a consequence of (3), in order for the vector-tubes of $\mathbf{c}$ to be of strength constant in time, it is necessary, but not sufficient, for the vector-tubes to be material tubes. The criterion (1) has not been used in the deduction of (2) or (3); thus even if the vector-tubes are of strength constant in time at each material cross-section they are not generally of equal strength at all cross-sections." [To the reviewer this conclusion is not clear. The expression "strength of a vector-tube" is clearly meaningful if the criterion (1) is fulfilled, being (for any $t$) a number associated with the tube. But the "strength at a material cross-section" (when not the same at all cross-sections) needs clarification. Also, it is not clear what is meant by the strength of a vector-tube being constant in time if the vector-tubes are not material tubes; if they are not material tubes, what tubes are compared at different times?] *J. L. Synge* (Dublin).

**Manarini, Mario. Sui paradossi di D'Alembert e di Brillouin nella dinamica dei fluidi.** Boll. Un. Mat. Ital. (3) 4, 352–353 (1949).

The author replaces the condition $v r^3 = 0$ in mean at infinity by the less restrictive condition $v r^2 = 0$ in mean at infinity in the direction of the axis of the motion in his extension of Brillouin's paradox. *L. M. Milne-Thomson.*

**Gerber, R. Sur la reduction à un principe variationnel des équations du mouvement d'un fluide visqueux incompressible.** Ann. Inst. Fourier Grenoble 1 (1949), 157–162 (1950).

Noting that previous attempts to obtain a variational principle equivalent to Navier's dynamical equations for viscous incompressible fluids have followed the Eulerian plan, the author proposes instead to let the volume of fluid under consideration be a material volume. For the rather special class of functions to be varied which he considers, he obtains a negative result. *C. Truesdell.*

**Magyar, F. Zur Ableitung des Croccoschen Wirbelsatzes.** Österreich. Ing.-Arch. 4, 138–140 (1950).

The author's objective is "to clarify the divergence of opinions with respect to the applications of Crocco's theorem." The "divergence" is to be found in recent German papers and lectures, and apparently arose from a generally incorrect form of the dynamical equations employed by Oswatitsch [Luftfahrtforschung 20, 260 (1943)]. The author bases his remarks upon the formula $2W \times V = T\nabla S - \nabla i_0$, where $2W$ is the vorticity, $V$ is the velocity, $T$ is the temperature, $S$ is the specific entropy, and $i_0$ is the stagnation enthalpy. This well-known relation was derived and fully discussed by A. Vazsonyi [see Quart. Appl. Math. 3, 29–37 (1945), §§ 6–8; these Rev. 7, 226]. *C. Truesdell.*

**Gilbarg, David, and Serrin, James. Free boundaries and jets in the theory of cavitation.** J. Math. Physics 29, 1–12 (1950).

The authors discuss the theory of a finite cavity containing a backward jet behind a body in two dimensional irrotational incompressible flow. The backward jet is a mathematical fiction whose departure from the physical facts does not appear to seriously effect the field of flow not in its immediate vicinity, according to experimental evidence now available. One of the usual conformal mapping techniques effective in solving problems involving straight boundaries is used to treat the problems of flow about an inclined plate and a wedge, in which, however, emphasis is placed on a generalization of the Schwarz transformation in which branch points are arbitrarily placed in the interior of the polygon which as a result occupies a Riemann surface. A set of conditions is given for the solution of flow problems involving curved boundaries. Techniques for satisfying these conditions are (understandably) absent. *E. Pinney.*

**Laitone, E. V. Approximate incompressible flow solutions for slender polygons.** J. Aeronaut. Sci. 17, 375–376 (1950).

**Heins, Albert E. Water waves over a channel of finite depth with a submerged plane barrier.** Canadian J. Math. 2, 210–222 (1950).

Using methods developed in an earlier paper [Amer. J. Math. 70, 730–748 (1948); these Rev. 10, 490], the author considers the problem of the title. The boundary value problem is reduced to a Wiener-Hopf integral equation which is solved explicitly. The reflection and transmission coefficients and phase shift angle for the barrier are also determined. *J. V. Wehausen* (Providence, R. I.).

**Havelock, T. H. The forces on a submerged spheroid moving in a circular path.** Proc. Roy. Soc. London. Ser. A. 201, 297–305 (1950).

Starting with an expression for the velocity potential for a source moving on a horizontal circular path beneath the

The reviewer apologizes for his failure to check the author's quotation of the work of Oswatitsch. He pointed out to the reviewer by Dr. Sidney D. Barrett, the author's eq. (1) is not to be found in Oswatitsch's paper where the equations and the analysis are perfectly correct. The "divergence of opinions" thus appears to be created by the author.

free surface of a heavy liquid, the author finds first both the tangential and radial force components on a sphere and then the tangential force and couple on a prolate spheroid, both moving in steady motion on circular paths. Computations of the tangential force were made for special cases, and the results are shown graphically and compared with the wave resistance in rectilinear motion.

*J. V. Wehausen* (Providence, R. I.).

**Dean, W. R. Note on the motion of liquid near a position of separation.** Proc. Cambridge Philos. Soc. **46**, 293–306 (1950).

This paper investigates the character of a two-dimensional motion, near the origin, of an incompressible viscous fluid. It is shown that, if the pressure $p$ is given as a power series in $x$ on the plane $y=0$ and the $x$-component $u$ of the velocity is given as a power series in $y$ on the line $x=0$, subject to the condition that the pressure gradient in $x$-direction balances the frictional force $u\partial^2u/\partial y^2$ at $y=0$, then the solution is uniquely determined for each of the cases: (a) skin friction is finite and nonvanishing at the origin and (b) skin friction is zero. In the latter case, the important terms of the stream-function in the neighborhood of the origin are of the form $A_3y^3+B_3y^2x$ and the origin is therefore a point of separation. The solution has been compared with that of S. Goldstein [Quart. J. Mech. Appl. Math. **1**, 43–69 (1948); these Rev. **10**, 270], by using boundary-layer theory.

*Y. H. Kuo* (Ithaca, N. Y.).

**Tournier, Marcel, et Bassière, Marc. Une solution des équations de la couche limite obtenue par la considération de phénomènes transitoires.** Recherche Aéronautique **1948**, no. 4, 67–72 (1948).

The authors study the boundary layer on a flat plate at zero angle of attack by means of approximate solution of Navier's dynamical equation. Their method is to substitute simple trial functions containing a few parameters and then to choose these parameters so that the dynamical equation is satisfied exactly at one point in the boundary layer and so that the boundary layer thickness coincides with that given by Blasius. Beginning by consideration of the transient solution obtained by Boussinesq [C. R. Acad. Sci. Paris **90**, 736–739 (1880)], the authors finally propose $u=V_0E_3(0.611yV_0^{1}/2(\nu x)^{1})$, where $E_3$ is an error function of third order. This formula departs from the result of Blasius by less than 3% everywhere, and has the advantage of being simple in form and easy to use. The results were summarized earlier [C. R. Acad. Sci. Paris **226**, 1124–1127 (1948); these Rev. **9**, 542].

*C. Truesdell.*

**Lessen, Martin. On the stability of the free laminar boundary layer between parallel streams.** Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1929**, 31 pp. (1949).

The stability of the flow associated with the laminar mixing of two parallel fluid streams with velocities $U_1$ and $U_2$ in the $x$ direction is discussed in the classical manner by investigating the behavior of solutions of the Orr-Sommerfeld equation [see, for example, Lin, Quart. Appl. Math. **3**, 117–142 (1945); these Rev. **7**, 225]. By choosing the perturbation stream function to be of the form $\phi(\eta)e^{i\alpha(x-ct)/\delta}$, where $\phi(\eta)$ gives the variation of the stream function normal to the main flow, $\delta$ is the width of the mixing zone, $\eta=y/(\nu x/U_1)^{1}$, $\alpha$ the real wave length and $c$ the complex phase velocity, it is clear that the flow is stable or unstable according to whether the imaginary part of $c$ is less than or greater than zero.

The Orr-Sommerfeld equation then follows from the Navier-Stokes equations

$$(f'-c)(\phi''-\alpha^2\phi)-f'''\phi=-(i/\alpha R)(\phi^{(4)}-2\alpha^2\phi''+\alpha^4\phi),$$

where $R$ is the Reynolds number based on the dimension $\delta$ and $f(\eta)$ is essentially the stream function of the laminar mixing process which satisfies the differential equation $ff''+2f'''=0$; the boundary conditions are $f'(\infty)=1$, $f'(-\infty)=U_2/U_1$, and $f(\eta_0)=0$ where $\eta_0$ represents the bounding streamline between the two flows. The condition that $\phi(\eta)$ be bounded throughout the fluid field admits two of the four solutions to the Orr-Sommerfeld equation, $\phi_1(\eta)$, $\phi_2(\eta)$, and establishes a characteristic value problem relating the parameters $\alpha$, $R$, and $c$, through the transcendental equation

$$F(\alpha,c,R)\equiv\begin{vmatrix}\phi_1'(-\infty)-\alpha\phi_1(-\infty) & \phi_2'(-\infty)-\alpha\phi_2(-\infty)\\ \phi_1'(\infty) \quad +\alpha\phi_1(\infty) & \phi_2'(\infty) \quad +\alpha\phi_2(\infty)\end{vmatrix}=0.$$

In order to investigate this relation, the solutions for the Orr-Sommerfeld equation were found by first obtaining asymptotic representations of the solution which satisfy the condition of boundedness and are valid for large moduli in the complex $\eta$ plane. Then employing these asymptotic solutions at finite values of $\eta$ as boundary values, the solutions $\phi_1(\eta)$ and $\phi_2(\eta)$ were continued into the mixing layer by numerical integration. The function $f(\eta)$ is likewise found numerically.

The transcendental relation is then investigated by choosing $\alpha$ and varying $c$ until the value of Reynolds number $R$ satisfying $f(\alpha,c,R)=0$ is real. By choosing $\Im(c)=0$, the corresponding pairs of numbers $(\alpha,R)$ define the neutral stability curve. The upper branch of this curve (i.e., for large $\alpha$) is found for the present problem and indicates much lower critical Reynolds numbers than occur, for example, in the case of the Blasius boundary layer. *F. E. Marble.*

**Markov, N. M. On the three-dimensional flow of a fluid through the stator and rotor blades of a turbine.** Doklady Akad. Nauk SSSR (N.S.) **71**, 245–248 (1950). (Russian)

The design of the profiles of the stator and rotor blades aims at securing a stable flow of fluid, and is based on the assumption of one-dimensional cylindrical flow. The equations of the flow are solved by using the condition that the stream does not lose energy in passing through the blades. This condition is well satisfied by the main part of the stream, but not in the boundary layers near the surfaces of the blades. The fluid in these layers is unstable, and either spreads toward the edges of the blade, or contracts toward the center of the blade. Thus the flow is three-dimensional and the assumption of a cylindrical one-dimensional flow does not hold in the boundary layers. The author derives and solves the partial differential equation for the variation of the direction of velocity in the three-dimensional boundary layer of a blade, using an approximation method. The formula obtained shows that the fluid in rotating slots may expand toward the lateral edges of a blade or move toward the center, depending on the rotational velocity; in the guiding slots of a turbine the fluid moves toward the center of a blade.

*T. Leser* (Lexington, Ky.).

**Taylor, G. I. The boundary layer in the converging nozzle of a swirl atomizer.** Quart. J. Mech. Appl. Math. **3**, 129–139 (1950).

The boundary layer along a conical surface immersed in a liquid swirling about the axis of the cone is studied. The flow inside the boundary layer is assumed to be axially

symmetrical with three velocity components. The simplified boundary layer equations are then solved approximately by Pohlhausen's method. The improved solution to this problem was given later by A. M. Binnie and D. P. Harris [see the following review]. *Y. H. Kuo* (Ithaca, N. Y.).

**Binnie, A. M., and Harris, D. P. The application of boundary-layer theory to swirling liquid flow through a nozzle.** Quart. J. Mech. Appl. Math. **3**, 89–106 (1950).

The boundary layer of a swirling laminar flow in a slightly convergent-divergent nozzle is studied for the purpose of estimating the reduction of discharge through a nozzle. In the first part of the paper, the pressure and the velocity in the core of the flow are obtained. With the aid of these results, the boundary-layer equations are derived in integral form by assuming three components in a spherical coordinate system and axial symmetry. The boundary layer thickness is finally calculated by Pohlhausen's well-known approximation and is found smaller than in either the case with swirl alone or with streaming alone.

*Y. H. Kuo* (Ithaca, N. Y.).

**Timman, R. Linearisation of the equations of two-dimensional subsonic compressible flow by means of complex characteristics.** Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 34**, i+14 pp. (1948).

*Timman, R. **Linearisation of the equations of two-dimensional subsonic compressible flow by means of complex characteristics.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 28–42.

The author derives the classical hodograph equation following the method of H. Lewy for transforming an equation to canonical form using characteristic coordinates. For subsonic flow, the characteristic directions are described by complex numbers. Approximate solutions are discussed by considering series developments in power series of $w/c_0$, valid for $(w/c_0)^2<2/(\gamma+1)$ ($w$=speed of fluid, $c_0$=stagnation velocity of sound, $\gamma$=ratio of specific heats). The first approximation gives incompressible flow, the second approximation corresponds to flow with $\gamma=-1$, an approximation already introduced by Chaplygin. The Chaplygin method of separating variables in the hodograph equation is introduced and flow around an object for $\gamma=-1$ is discussed, following previous work by Lin [Quart. Appl. Math. **4**, 291–297 (1946); these Rev. **8**, 418]. *P. A. Lagerstrom.*

**Ludloff, H. F., and Reiche, F. Linearized treatment of supersonic flow through ducts.** J. Aeronaut. Sci. **16**, 5–21 (1949).

Linearized steady axisymmetric flows through sharp lipped ducts have been constructed by a modification of the Karman-Moore method [Trans. A.S.M.E. **54**, 303–310 (1932)] for calculating linearized supersonic flows about thin pointed bodies of revolution. The flows are produced by superposition of a uniform flow and that due to a source or sink distribution on a semi-infinite circular cylinder with intensities $f(\xi)=A'\xi+A$, where $\xi$ is axial distance from the cylinder's lip, and $A$ and $A'$ are constants. Velocity components, potential and stream functions, and pressure coefficients are obtained from power series expansions, explicit expressions in terms of elliptic integrals, or graphical integration. Three cases are considered. (1) $A'=0$ yields ducts which contain throats, have no wave drag, but have infinite velocities on the downstream nappe of the Mach cone whose upstream nappe passes through the lip. (2) $A=0$ yields ducts with continuous velocity fields but with wave drags and no throats. (3) $f(\xi)=A'\xi$ for $0\leqq\xi\leqq B$ (=constant), $f(\xi)=A'B$ for $\xi>B$ yields ducts with throats, continuous velocity fields, and no wave drags. Flows of type (3) are produced by superposition of flows of type (2). By means of recurrence formulae, calculations of type (2) can be based partially on results obtained for type (1). *J. H. Giese.*

**Vavra, M. H. Steady flow of nonviscous elastic fluids in axially symmetric channels.** J. Aeronaut. Sci. **17**, 149–156, 172 (1950).

The differential equations for the steady flow of a nonviscous, compressible fluid in an axially symmetric channel are set up using as coordinates the angle $\theta$ between a fixed and a variable meridian plane and the intrinsic coordinates $s$ and $n$ of the projection of a streamline upon the meridian plane. If $V_1$ and $V_2$ denote respectively the velocity components perpendicular and parallel to the meridian plane, they are quantities independent of $\theta$ that are found to satisfy the equation $\partial V_2^2/\partial n+2R^{-1}V_2^2+2r^{-1}V_1\partial(rV_1)/\partial n=0$, where $R$ is the radius of curvature of the meridian streamline projection and $r$ is the distance from the axis of symmetry. This differential equation is solved for $V_2^2$ on the assumption that $R$ and $V_1$ are known functions of $n$. Actually these functions need be given only along a single normal curve if the rate of mass flow is specified, for then $R$ and $V_1$ may be found for an arbitrary normal curve. The method is illustrated by an example of an irrotational flow and also one of a rotational flow. The method has applications to the design of inlet ducts and diffusers. *M. Marden.*

**Souriau, Jean-Marie, et Chastenet de Géry, Jérôme. Extension de la méthode de Küssner aux profils épais.** C. R. Acad. Sci. Paris **230**, 1828–1830 (1950).

The unsteady motion of an airfoil of finite thickness in two-dimensional, incompressible flow is treated by superimposing on the exact solution (nonlinear boundary condition) for steady flow over the profile an unsteady flow perturbation and linearizing with respect to the amplitude of the latter. The resulting problem for the perturbation potential then may be solved by known methods [Küssner, Luftfahrtforschung **17**, 370–378 (1940); Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **979** (1941); these Rev. **2**, 330, 331]. *J. W. Miles* (Los Angeles, Calif.).

**Jones, Robert T. Properties of low-aspect-ratio pointed wings at speeds below and above the speed of sound.** Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **835**, 5 pp. (1946).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1032** (1946); these Rev. **9**, 114.

**Lomax, Harvard, and Heaslet, Max. A. Linearized lifting-surface theory for swept-back wings with slender plan forms.** Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1992**, 41 pp. (1949).

The flows studied here are those for which the first term of the linearized small-perturbation potential equation (1) $(1-M^2)\phi_{xx}+\phi_{yy}+\phi_{zz}=0$ may be neglected. In (1), $M$ is the stream Mach number, and $x$ is measured in the stream direction. A "slender" wing is one for which this approximation is valid, and it is proposed that the criterion is that $|1-M^2|^{\frac{1}{2}}\times$(span) be small compared to the length of the wing in the streamwise direction. Classical methods are employed to construct solutions to the two-dimensional

Laplace equation that results. Detailed analyses are then carried out for a class of flat wings having straight, sweptback leading edges. There is difficulty in satisfying the trailing-edge conditions when the shape is given arbitrarily. A special case is treated, for which there is no trailing-vortex wake between the two sweptback trailing edges. The shape of the trailing edge for this case is worked out and plotted. The span loading and lift are calculated for three subcases, and the qualitative nature of the pressure distributions is also discussed. For a triangular wing the results agree with R. T. Jones's, which were based on an equivalent approximation [see the preceding title]. For the other two cases a comparison can be made with D. Cohen's results based on equation (1) [same Tech. Notes, no. **1555** (1948); these Rev. **9**, 478]; the principal difference seems to be in the position of discontinuities of loading. The report closes with a calculation of vortex drag (induced drag). Some results of the present analysis were given, without the detailed treatment, in a previous paper [same Tech. Notes, no. **1824** (1949); these Rev. **10**, 643]. *W. R. Sears.*

**Bilharz, Herbert. Über eine Anwendung des Dirichletschen Diskontinuitätsfaktors in der Strömungslehre.** Arch. Math. **2**, 27–32 (1 plate) (1949).

The integral equation of Prandtl's lifting line theory is solved in two special cases. In the first problem the vorticity distribution along the $y$-axis is found which makes the angle of attack the following step function: $\alpha(y)=2\delta$ for $0\leqq|y|<l$, $\alpha(y)=\delta$ for $|y|=l$, $\alpha(y)=0$ for $|y|>l$. The integral equation is solved by Fourier analysis. In the second problem $\alpha$ is also a simple step function which is odd in $y$. Graphs of the vorticity for several values of the parameters of the step function are presented. It should be noted that the wings considered are three-dimensional but of infinite span. The angle of attack varies spanwise (i.e., in the $y$-direction) but is prescribed for all values of $y$.

*P. A. Lagerstrom* (Pasadena, Calif.).

**Goodman, Theodore R. The problem of edges in the doublet distribution method of obtaining supersonic lift.** J. Aeronaut. Sci. **17**, 376–378 (1950).

**Stewartson, K. Supersonic flow over an inclined wing of zero aspect ratio.** Proc. Cambridge Philos. Soc. **46**, 307–315 (1950).

The boundary-value problem indicated by the title is formulated by means of the Laplace transform. This results in an integral equation in one variable, which Schwarzschild [Math. Ann. **55**, 177–247 (1902)] solved for large $p$, corresponding to the leading-edge region in this problem, by successive approximations. This is the solution applied to the airfoil problem by Gunn [Philos. Trans. Roy. Soc. London. Ser. A. **240**, 327–373 (1947); these Rev. **9**, 313]. It becomes unwieldy and inaccurate for distances greater than a few spans behind the leading edge. Here the author finds an approximation for small $p$, i.e., for large chordwise distances, by the familiar method of expansion in terms like $p^n(\ln p)^m$. He is able to compute total lift and center of pressure easily; the latter lies at the leading edge. Finally he interprets the Laplace transforms in the series approximating the pressure, term-by-term. The resulting function, evaluated along the centerline of the wing, is in good agreement with Gunn's results, even for rather small chordwise distances. *W. R. Sears* (Ithaca, N. Y.).

**Batchelor, G. K. On the spontaneous magnetic field in a conducting liquid in turbulent motion.** Proc. Roy. Soc. London. Ser. A. **201**, 405–416 (1950).

In this paper the interaction between the electromagnetic and the hydrodynamic forces in an incompressible conducting fluid in turbulent motion is considered. It is shown that in the limit of very high electrical conductivity of the fluid, the equations of motion take the form:

$$(1)\quad \frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_k}\left(u_iu_k-\frac{\mu}{4\pi\rho}H_iH_k\right)=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p+\frac{\mu}{8\pi}|\mathbf{H}|^2\right)+\nu\nabla^2u_i,$$

and

$$(2)\quad \frac{\partial H_i}{\partial t}+\frac{\partial}{\partial x_k}(H_iu_k-H_ku_i)=\lambda\nabla^2H_i,$$

where $u_i$ and $H_i$, $i=1, 2, 3$, denote the components of the velocity and the magnetic field, $\rho$ is the density, $p$ is the pressure, $\mu$ is the magnetic permeability, $\sigma$ is the electrical conductivity, $\nu$ is the kinematic viscosity and $\lambda=1/4\pi\mu\sigma$. [In the foregoing equations summation over repeated indices is to be understood.] Equations (1) and (2) should be supplemented by the conditions $\operatorname{div}\mathbf{u}=\operatorname{div}\mathbf{H}=0$.

The author observes that the equation governing $H_i$ is identical in form with the equation governing the vorticity in an incompressible fluid in the absence of electromagnetic forces. From this analogy between vorticity and magnetic field it is deduced that the lines of magnetic force move with the fluid when the conductivity is infinite and that the small scale components of turbulence have the more powerful effect on the magnetic field. The stability of a purely hydrodynamic system to small disturbing magnetic fields is considered, and it is shown that the magnetic energy of the disturbance will increase provided $\lambda<\nu$. When this inequality is satisfied the rate of growth of the energy in the magnetic field will be exponential in the first instance. The time constant for this rate is estimated at $(\nu/\epsilon)^{\frac{1}{2}}$ where $\epsilon$ is the rate of dissipation of kinetic energy by viscosity per unit mass of the fluid. If $\lambda<\nu$ we may expect that eventually a steady state will be established and the author suggests (again arguing from the analogy between the vorticity and the magnetic field) that the energy in the magnetic field will reside principally in the small scale components of the turbulence. Estimates of the amount of energy in the magnetic field and the region of the spectrum in which it will lie are given in terms of the observable properties of turbulence.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Bass, J. Les méthodes modernes du calcul des probabilités et leur application au problème de la turbulence.** Groupement Français Dévelop. Rech. Aéronaut. Rap. Tech. no. **28**, 241 pp. (1946).

**Litwiniszyn, Jerzy. Stationary flows in heterogeneously unisotropic mediums.** Ann. Soc. Polon. Math. **22** (1949), 185–194 (1950).

Guided by analogy to the theories of diffusion of heat or electric current, the author sets up the following equation for flow of an incompressible fluid in a porous medium: $V^i=A^{ij}p_{,i}$, $V^i{}_{,j}=0$, or, equivalently, $(A^{ij}p_{,i})_{,j}=0$, where $V^i$ is the diffusion velocity, $A^{ij}$ is the permeability tensor, and $p$ is the pressure drop. He then obtains a Fourier series solution for the problem of flow in an isotropic semi-infinite strip along whose finite edge the pressure drop is linear,

along whose infinite edges no flow takes place, and whose permeability falls off exponentially as a function of the distance from the finite edge. *C. Truesdell.*

**Tartakovskiĭ, B. D. On the theory of the propagation of plane waves in homogeneous layers.** Doklady Akad. Nauk SSSR (N.S.) **71**, 465–468 (1950). (Russian)

Le calcul des coefficients de réflexion et de réfraction au passage des ondes planes à travers $n$ couches homogènes planes se réduit à la résolution d'un système de $2(n+1)$ équations algébriques. On obtient ces équations en mettant dans les conditions aux limites les expressions des potentiels des ondes. Les résistances au passage des ondes et les épaisseurs des couches sont données. L'auteur examine le cas des ondes acoustiques dans les couches liquides caractérisées par leurs densités, vitesses de propagation des ondes acoustiques et épaisseurs. Il construit les déterminants du système et cherche les voies les plus simples pour les calculs. Dans deux cas particuliers les calculs sont exécutés jusqu'au bout. *V. A. Kostitzin* (Paris).

**Martyn, D. F. Cellular atmospheric waves in the ionosphere and troposphere.** Proc. Roy. Soc. London. Ser. A. **201**, 216–234 (1950).

The author considers small wave disturbances in an atmosphere of several layers with linear temperature and wind distribution in the vertical. The wave lengths are assumed small enough so that the earth's rotation can be neglected. Special attention is given to the duct effects of change in the temperature distribution. It is found that a bounded duct is formed if a ground inversion is followed by a normal temperature lapse rate, if the lapse rate increases, and if the temperature increases at a point either discontinuously or continuously. In these cases cellular waves may be set up in the duct, and the microbarometric oscillations observed at the ground are attributed to these cellular waves rather than Helmholtz waves at a discontinuity. It is demonstrated that these cellular waves must have frequencies in the vicinity of those for the vertical oscillations of an isolated air parcel. To explain the observed behaviour of the travelling disturbances in the ionosphere the geomagnetic field has to be taken into account. This produces a phase change with height in these oscillations, simulating vertically moving electron clouds. It appears that the boundary conditions in the $F_2$ layer require for the ratio of the specific heats a value considerably less than 1.4, the value for air in the lower atmosphere. *B. Haurwitz* (Cambridge, Mass.).

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### *Elasticity, Plasticity*

*Sommerfeld, Arnold. **Mechanics of Deformable Bodies. Lectures on Theoretical Physics, Vol. II. Translated from the second German edition by G. Kuerti.** Academic Press, Inc., New York, N. Y., 1950. xi+396 pp. $6.60.

The first German edition [1945] of this book was reviewed in these Rev. **8**, 356. In the second German edition [1947] a supplement on tensor calculus was added and the section on turbulence was revised on the basis of recent work by von Weizsäcker and Heisenberg. In the present translation several footnotes concerning questions of terminology have been added, and references to well-known American and British texts have been made to replace similar references to German texts. *W. Prager.*

**Milne Thomson, L. M. Plane elastic problems. I.** Revista Mat. Hisp.-Amer. (4) **9**, 110–123 (1949). (Spanish)
**Milne Thomson, L. M. Plane elastic problems. II.** Revista Mat. Hisp.-Amer. (4) **9**, 141–153 (1949). (Spanish)

Dans la première partie l'auteur présente une solution très générale des équations de l'équilibre élastique plan, en s'aidant de la représentation complexe des coordonnées cartésiennes, pour la mettre ensuite en relation avec un potentiel des forces agissant dans le plan. Dans la deuxième partie on considère différents cas particuliers: le plan indéfini, le semiplan, le cercle, en rappelant de même rapidement l'application de la transformation conforme pour passer du contour circulaire à des formes plus complexes. Un grand nombre de fautes dans les formules, qu'il est difficile d'attribuer toutes au compositeur, oblige le lecteur à un travail d'interprétation. *B. Levi* (Rosario).

**Fichera, Gaetano. Sui problemi analitici dell'elasticità piana.** Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 1–22 (1949).

The author is concerned with the existence and uniqueness of solution of boundary value problems in plane elasticity. He formulates these problems in a fashion slightly more general than is customary, so that they shall contain as special cases classical problems of potential theory. Beginning his argument with reciprocity formulae and Green's formulae, he proceeds by analogy to the methods of potential theory, treated from the viewpoint of linear functional analysis. Thirty-three theorems are stated, and an indication of proof for some of them is supplied. Among them is one exhibiting a set of homogeneous polynomial solutions and another stating that these form a complete system. *C. Truesdell* (Bloomington, Ind.).

**Fichera, Gaetano. Sul calcolo delle deformazioni, dotate di simmetria assiale, di uno strato sferico elastico.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 582–589 (1949).

L'auteur étudie, grâce à une méthode de Picone, la déformation d'un corps homogène élastique isotrope soumis à des forces de masses ou superficielles, mais surtout le cas d'une couche sphérique sous forces de masses. On suppose que la déformation est douée de symétrie axiale (vecteur situé dans un plan méridien et par rapport à lui indépendant de la longitude). On donne des développements en série des composantes de ce vecteur et on s'en sert pour résoudre le problème correspondant à des forces à la frontière présentant aussi la symétrie axiale. Le cas du corps pesant avec pression hydrostatique se résoud en termes finis. *M. Brelot.*

**Melan, E. Wärmespannungen in Scheiben.** Österreich. Ing.-Arch. **4**, 153–156 (1950).

An isotropic stress free body in which there is a temperature distribution is considered. By the introduction of a heat displacement potential function $\theta$ into equations arising from the equilibrium conditions and Hooke's law it is shown that the stresses and strains may be expressed in terms of $\theta$. By use of an Airy stress function it is possible to make all the stresses except $\sigma_{zz}$ vanish. As a special case, no stresses occur in a thin disc subjected to a stationary temperature distribution as long as there are no sources of heat in the interior of the disc. *C. G. Maple* (Washington, D. C.).

**Berman, M. È. Concerning the center of flexure.** Doklady Akad. Nauk SSSR (N.S.) **72**, 27–30 (1950). (Russian)

By a straightforward derivation the author expresses the coordinates of the center of flexure as line integrals of the

harmonic function of torsion. Contrary to the author's claim, the method is well known [cf., for example, Sokolnikoff and Specht, Mathematical Theory of Elasticity, McGraw-Hill, New York-London, 1946]; however, his final expressions seem to be somewhat easier to compute.

*H. I. Ansoff* (Santa Monica, Calif.).

**Rašković, Danilo. A singularity of the bending function of beams with rectangular cross-section.** Glas Srpske Akad. Nauka **195**, 79–87 (1949). (Serbian)

**Tolotti, Carlo, e Grioli, Giuseppe. Sul calcolo delle piastre con nervature.** Giornale del Genio Civile, no. 6 = Pubbl. Ist. Appl. Calcolo no. **258**, 36 pp. (1949).

The authors' objective is to formulate a general and reasonably accurate theory of the deformation of plates supported by straight stiffeners. Treating the stiffeners first as beams of uniform section which are not attached to the plate but which exert upon it concentrated loads arising from their deformation, the authors obtain a complicated linear integro-differential equation satisfied by the displacement of the plate. They then obtain a general solution for this equation in the form of a series in whose terms occur the coefficients of the proper-function expansion of the Green's function for the plate itself. Any finite approximation to this solution may be obtained by solving a system of linear algebraic equations. A virtue of the method is that the number of these equations does not depend upon the number of stiffening members, although, of course, increasing the number of members renders the equations themselves more complicated. The authors then take up the problem of a plate stiffened by attached beams. They first treat a single T-beam by the method of beam theory, and they regard the stiffened plate as made up of a system of such beams. Thus they obtain an integro-differential equation identical in form with that valid for the case when the stiffeners are not attached, except that the constants therein have different values. This process involves selecting an effective width $b^*$ of the interacting portion of the plate. The authors then apply their result to various simple types of stiffened rectangular plates, giving explicit formulae for the first and second approximations and then tables of numerical values. They find that the values of the bending moments are insensitive to choice of $b^*$, within reasonable limits.

*C. Truesdell* (Bloomington, Ind.).

**Jaramillo, T. J. Deflections and moments due to a concentrated load on a cantilever plate of infinite length.** J. Appl. Mech. **17**, 67–72 (1950).

**Green, A. E., and Zerna, W. The equilibrium of thin elastic shells.** Quart. J. Mech. Appl. Math. **3**, 9–22 (1950).

The authors begin with a brief description of some of the existing theories of shells. They then give a concise and clear development of the general theory of infinitesimal strain and of stress-resultants in terms of variables appropriate to shell theory. They use a tensorial treatment and for the geometry of strain they employ dimensionless variables. To obtain equilibrium equations for the resultants they integrate the three-dimensional equations in an appropriate coordinate system. [This method was used by the reviewer [Trans. Amer. Math. Soc. **58**, 96–166 (1945); these Rev. **7**, 231]; see §§ 4, 8, where it was shown that the authors' neglect of volume forces is unnecessary.] After these general preliminaries they introduce approximations. Quantities multiplied by the ratio of the thickness of the shell to a characteristic length are neglected in many expressions but retained in some others. The resulting theory is formulated in terms of five quantities: the three displacements averaged along the normal to the shell and the two tangential displacements similarly averaged after weighting by the normal distance. Thus five boundary conditions (not formulated by the authors) can be satisfied.

*C. Truesdell* (Bloomington, Ind.).

**Parkus, H. Die Grundgleichungen der Schalentheorie in allgemeinen Koordinaten.** Österreich. Ing.-Arch. **4**, 160–174 (1950).

The author derives equations for the small deformations of thin shells, using tensorial methods. He states that his method is different from those of five recent writers; however, in all respects but two it is a special case of that given by Synge and Chien [pp. 103–120, von Kármán Anniversary Volume, California Institute of Technology, Pasadena, Calif., 1941; these Rev. **3**, 30, 371] where all quantities are developed in power series in the normal distance from the reference surface. The first point of difference is that the author uses stress-strain relations containing a thermal dilatation term; the temperature is also expressed as a power series. The second is that the author develops his equilibrium equations from a variational principle so as to obtain boundary conditions as well. In this way a Kirchhoff equivalent edge stress appears. The author actually writes down the results for the case when terms in the square of the normal distance from the middle surface are retained. Since he makes the usual assumptions of the Love theory, however, it is likely that the retention of the quadratic terms decreases rather than increases the accuracy of the results. In fact the author notes several inconsistencies among his various equations but regards them as negligible. At the end of the paper the simplifications resulting for some particular shell shapes are noted.

*C. Truesdell.*

**Alumyaè, N. A. A variational method for the investigation of thin elastic shells in the buckling stage.** Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 197–202 (1950). (Russian)

The buckling of thin elastic shells is discussed under the condition of hydrostatic surface loading. A variational formula is presented in which the associated contour integral contains a variation of the stress function instead of a variation of the tangential component of deflection which is employed in Lagrange's formula. The choice of admissible functions is thus simplified. The results are applied to the problem of buckling of a cylindrical strip.

*H. I. Ansoff* (Santa Monica, Calif.).

**Alumyaè, N. A. On a formula for the critical stress of a momentless stressed state of thin-walled elastic shells.** Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 647–649 (1949). (Russian)

The critical stress formula for (1) a spherical shell compressed by a distributed load, (2) a centrally compressed cylindrical shell, and (3) a centrally compressed conical shell, is well known. The author derives a general critical stress formula for a shell of an arbitrary shape, which reduces to the above mentioned cases for suitable boundary conditions. The author points out that his results have mainly a qualitative character and, with the exception of the simplest cases, cannot be used for exact computations.

*T. Leser* (Lexington, Ky.).

**Cicala, Placido. Sul comportamento elastico di una parete sottile quasi cilindrica.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 99–103 (1949).

An approximate solution for certain cases of the elastic buckling of cylindrical shells is given; the results are presented in diagrams. *P. F. Neményi* (Washington, D. C.).

**Federhofer, K. Kippsicherheit des kreisförmig gekrümmten Trägers mit einfach-symmetrischem, dünnwandigem und offenem Querschnitte bei gleichmässiger Radialbelastung.** Österreich. Ing.-Arch. **4**, 27–44 (1950).

The differential equations governing the lateral, small-deflection buckling of a circular arch are developed, presuming that the cross-section of the arch is thin-walled and open, and is symmetrical about the plane of the arch. The external loading consists of a uniform radial pressure, and the lateral deflections and twists are opposed by an elastic restraining-medium. Solutions are then obtained for the buckling of complete rings, as well as arches with various end-constraints. It is also shown how the present analysis is applicable to determination of the natural vibrational frequencies of the system. The present treatment differs from those already published by virtue of the considerations relating to the thin-walled, open cross-section.

*M. Goland* (Kansas City, Mo.).

**Arutyunyan, N. H. On statically indeterminate systems with supports which settle.** Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 489–500 (1949). (Russian)

In the theory of indeterminate structures, the settling of supports is usually treated in one of the following two ways: the displacement of the support is either assumed to be known beforehand or it is assumed to be proportional to the reaction at the support. In the present paper, these assumptions are replaced by a more careful evaluation of the support displacement by methods of soil mechanics.

*W. Prager* (Providence, R. I.).

***Barrois, W., et Simon-Suisse, J. Calcul des charges de flambage au moyen des matrices de coefficients d'influence.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 52–62.

***Hetényi, M. A general solution for the bending of beams on an elastic foundation of arbitrary continuity.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 229–237.

***Naleszkiewicz, J. On the cooperation of two cantilever spars with a shear-resisting skin.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 214–228.

**Takeyama, Hisao. On the bending strength and vibrations of radially rotating bars. II.** J. Soc. Appl. Mech. Japan **2**, 91–93, 110 (1949). (Japanese. English summary)

[For part I cf. the same J. **1**, 121–129 (1948); these Rev. **10**, 655.] The deflection of a rotating bar like a turbine blade, acted on by a single load, is obtained analytically; the formula is suitable for numerical and graphical integrations. *From the author's summary.*

**Takeyama, Hisao. On the bending strength and vibrations of radially rotating bars. III.** J. Soc. Appl. Mech. Japan **2**, 94–96, 110 (1949). (Japanese. English summary)

[Cf. the preceding review.] When the bar is hinged at the end near the rotating axis, its deflection caused by a concentrated load is found to have singularities at zero angular velocity. Here it is made clear that these singularities are apparent and have no effect on the solutions of the integral equations of the vibrations.

*From the author's summary.*

**Vogel, Th. Vibrations d'une poutre continue sous l'effet d'une charge mobile.** Ann. Ponts Chaussées **1949** (119e année), 407–424 (1949).

**Rydlewski, St. Vibrations propres d'une poutre à treillis aux noeuds rigides.** Arch. Méc. Appl., Gdańsk **1**, 99–119 (1949).

The author considers the vibration of plane trusses with rigid joints, all motions being in the plane of the truss. The elementary theory of the transverse and longitudinal vibrations of a single bar is applied to a general member of the truss. This yields expressions for the forces and couples acting on the ends of the general member in terms of the displacements and rotations of the ends of the member. The three equations of equilibrium of each joint of the truss, which involve these forces and couples, can then be expressed as a set of equations linear in the displacements and rotations of the joints. The determinant of this set of equations is equated to zero to yield an equation for the determination of the fundamental frequency of the truss. This equation is solved graphically. *G. E. Hay.*

**Itokawa, Hideo. A new method of solving the problem of the vibration of plates.** J. Soc. Appl. Mech. Japan **2**, 96–98, 110 (1949). (Japanese. English summary)

The general solution of the equation of the vibration of plates is formed by a series of Bessel functions and circular functions in polar coordinates. Frequencies of a rectangular plate calculated by this method showed good results, and frequencies of a triangular plate were calculated anew.

*Author's summary.*

**Kuskov, A. M. The diffraction of elastic stable oscillations.** Doklady Akad. Nauk SSSR (N.S.) **70**, 197–200 (1950). (Russian)

The three dimensional boundary value problem of dynamical elasticity for an infinite homogeneous isotropic medium exterior to a closed surface $S$ on which the stresses are given is equivalent to the determination of a scalar potential function and a vector potential function, each of which satisfies the wave equation where the two constants are different and each expressed in terms of Lamé's constants and the frequency. An appropriate radiation condition is assumed and the system is reduced to the solution of a system of singular Fredholm integral equations of the type solved by S. G. Mihlin [Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 29–112 (1948); these Rev. **10**, 305].

*C. G. Maple* (Washington, D. C.).

**Thomson, William T. Transmission of elastic waves through a stratified solid medium.** J. Appl. Phys. **21**, 89–93 (1950).

L'auteur étudie la transmission d'une onde élastique plane, arrivant sous un angle quelconque, à travers un milieu stratifié composé des couches parallèles possédant des caractéristiques physiques et mécaniques différentes. Le problème se réduit aux équations du type à peu près classique. Les efforts dans le milieu dépendent naturellement de la dilatation et de la distribution des "tourbillons" dans le milieu. La méthode des matrices permet de résoudre

les équations en utilisant les formules bien connues d'inversions. L'auteur termine par l'examen de quelques cas particuliers. *V. A. Kostitzin* (Paris).

**Oldroyd, J. G. On the formulation of rheological equations of state.** Proc. Roy. Soc. London. Ser. A. **200**, 523–541 (1950).

This paper discusses the general form of the equations of state, and how they must be combined with the equations of motion and continuity to determine stress and strain distributions. The equations of state are considered in terms of a convected coordinate system moving with the material, and involve tensors associated with a particular material element, independent of its motion as a whole in space or of the states of neighboring elements. The kinematics of an element is expressed by the variation of the metric tensor, the force system by the stress tensor. The equations of state may include derivatives and integrals with respect to time and the history represented by values throughout previous periods of time. The transformation to a fixed coordinate system is considered for the purpose of combining the equations of state with the equations of motion and continuity. Examples of application of the theory are given.

*E. H. Lee* (Providence, R. I.).

**Kondo, Kazuo. A proposal of a new theory concerning the yielding of materials based on Riemannian geometry. II.** J. Jap. Soc. Appl. Mech. **2**, 146–151 (1949).

[For part I, cf. the same vol., 29–31 (1949); these Rev. 11, 282. In the original, the author's name is misprinted Kanzo Kondo.] The sharp yield point of a material such as mild steel can be regarded as a critical point of instability, at which the material manifold is suddenly bent, as it were, incompatibly forming a three-dimensional Riemannian manifold. We have a sort of bending force $-N^{\lambda}=\Omega^{\lambda}_{ij}\sigma^{ij}$ in directions normal to the material in the enveloping space, where $\Omega^{\lambda}_{ij}$ denotes the Euler-Schouten curvature tensor and $\sigma^{ij}$ the ordinary stress tensor. It must be balanced by the resistance of the material, introduced by the author, taking into consideration that the ordinary laws of the theory of elasticity are not valid for phenomena of a scale comparable with the microscopic structure of the material. As the field equation of equilibrium at the yield point the author obtains $B\Delta\Delta w+\sigma^{ij}\partial^2 w/\partial x^i\partial x^j=0$ for the case of isotropy. The author intends to discuss the boundary conditions in forthcoming papers. *A. Kawaguchi* (Sapporo).

**Majer, J. Beitrag zu den dreiachsigen Spannungs-Dehnungs-Beziehungen fester Stoffe.** Österreich. Ing.-Arch. **4**, 140–153 (1950).

The stress-strain law proposed in this paper lets the permanent strain in solids be proportional to the stress deviation. The factor of proportionality is the sum of two invariants of the stress tensor; the first of these is supposed to represent the intercrystalline deformation, and the second the intracrystalline deformation. The author does not explain how such a law is to represent the known dependence of the permanent strain on the history of loading.

*W. Prager* (Providence, R. I.).

**Il'yušin, A. A. The theory of elastic-plastic deformation and its applications.** Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 769–788 (1948). (Russian)

This paper contains a brief summary of the theory and a somewhat lengthy list of applications covering a good deal of the author's previous work on the subject. The results are given without derivation and unfortunately without reference to previous papers. Various statements leave the reader with the false impression that these results, based on a simple deformation theory of plasticity, are the same as would be obtained by use of the less restricted and more involved flow theories. While it is true, as the author states, that the answers must agree in cases of "simple loading" (wherein the components of the stress tensor remain in a fixed proportion at each point), the examples chosen for the most part do not lead to simple loading.

*H. J. Greenberg* (Pittsburgh, Pa.).

**Southwell, R. V. On the computation of strain and displacement in a prism plastically strained by torsion.** Quart. J. Mech. Appl. Math. **2**, 385–397 (1949).

The relaxation solutions of F. S. Shaw are extended to illustrate that only moderate plastic strains are needed to greatly relieve elastic stress-concentrations. A Prandtl-Reuss stress-strain relation is used to find the warping and the boundary of the elastic plastic region for a hollow rectangle. Reference is made to the independent work of P. Hodge [J. Appl. Mech. **16**, 399–405 (1949); these Rev. **11**, 559] in which relaxation is not employed.

*D. C. Drucker* (Providence, R. I.).

**Allen, D. N. de G., and Southwell, Richard. Relaxation methods applied to engineering problems. XIV. Plastic straining in two-dimensional stress-systems.** Philos. Trans. Roy. Soc. London. Ser. A. **242**, 379–414 (1950).

The specific problems treated in this paper by relaxation methods concern tensile specimens with semicircular and V notches. In each case the growth of the plastic region is studied under the assumption of plane strain or plane stress. In most cases the plastic regions originate at the bottoms of the notch and grow from there until they merge on the center line of the specimen. In the case of the specimen with semicircular notches in plane strain, however, new plastic regions start to develop from points on the center line after the plastic regions originating at the bottoms of the notches have grown to a certain size. To this reviewer's knowledge, the possibility of this type of behavior had been completely overlooked in the literature. *W. Prager.*

**Jacobs, J. A. Relaxation methods applied to problems of plastic flow. I. Notched bar under tension.** Philos. Mag. (7) **41**, 349–361 (1950).

The relaxation technique using inexact patterns for approximate values is employed to solve elastic-plastic problems of plane strain [modification of the paper reviewed above]. An incremental stress-strain relation is used (Prandtl-Reuss). Curves of the Mises equivalent stress are drawn for a symmetrically slit tension specimen with ratios of gross to net section of 8, 4, and 2. As in the Allen and Southwell solutions, plastic enclaves start at the root of the notch, spread out and go toward the longitudinal center line at a considerable distance from the transverse center line. A plastic enclave starts from a relatively far-away point at the longitudinal center line and spreads to meet the enclave from the root only for the section ratio of 8. For smaller ratios the enclaves from the root simply meet at the point on the axis of the bar. It should be emphasized, as the author does, that his solutions when the plastic zone completely stretches across the bar are not necessarily the stage of unrestricted plastic flow. The diagrams therefore do not represent "final results" but only the elastic-plastic configuration at a given load. *D. C. Drucker.*

**Jacobs, J. A. Relaxation methods applied to problems of plastic flow. II.** Philos. Mag. (7) **41**, 458–467 (1950).

No additional theory appears in this part [see the preceding review], but elastic-plastic solutions are obtained for a sharp ended bar under an equilibrium system of end compression and shear and for a block compressed between two smooth plates which are much shorter than the block. In the latter case the stage when a plastic zone spreads across the block seems to the reviewer to require a total load which is only a small fraction of the unrestricted flow or collapse load. Further development of the solution will not be simple as the author indicates that the determination of the very rapidly changing elastic-plastic boundaries becomes very difficult at this time. *D. C. Drucker* (Providence, R. I.).

**Lin, Tung-Hua. Inelastic column buckling.** J. Aeronaut. Sci. **17**, 159–172 (1950).

A successive approximation procedure is given for calculating the complete history of the deflection curve of an initially curved pin-ended column in the plastic range. Reversal of strain, the Shanley concept, and maximum possible load are discussed in detail and many computations and curves are given. A modified or elastically equivalent section is employed to reduce the analysis to the elastic case, although with a variable modulus in space and in time. The reviewer was unable to understand the appearance in the basic differential equation of the distance $x_0$ between the geometric centroid and the neutral axis for bending in the moment increment term $\Delta P(y+x_0)$ and its absence in the curvature increment $(d^2/dx^2)\Delta y$. *D. C. Drucker.*

## MATHEMATICAL PHYSICS

### *Optics, Electromagnetic Theory*

**Platzeck, Ricardo. Theory of the geometric aberrations of centered optical systems.** Revista Unión Mat. Argentina **14**, 182–196 (1950). (Spanish. English summary)

This paper gives a lucid summary of the two important methods of treating optical systems, the reduction of all the aberration coefficients to the coefficients of the angle eiconal and the direct method of optics. The author develops this method as a generalization of formulae by Sampson, who investigated the first order aberrations along a ray with the help of matrix equations. He does not mention the elaborate previous work in the field carried on by Boegehold, T. Smith, and the reviewer. The results can be simplified considerably by introducing $m_0$ and $m_i$ as coordinates instead of $p_0$ and $p_i$. In the second part, the coefficients of the transformation matrix are not independent but subject to certain equalities connecting their first order differential quotients. [Cf. Herzberger, Trans. Amer. Math. Soc. **53**, 218–229 (1943); these Rev. **4**, 204.]

*M. Herzberger* (Rochester, N. Y.).

**Scandone, F. Calcolo della trasmissione e riflessione nelle lamine multiple sottili.** Ottica (N.S.) **3**, 53–60 (1949).

Nel caso d'incidenza normale su uno strato di $k-1$ lamine parallele non assorbenti, scrivendo le condizioni di continuità per le componenti tangenziali del campo, si trova che l'onda progressiva (+) e l'onda regressiva (−) nel primo e nell'ultimo mezzo sono legate dall'equazione

$$\begin{pmatrix} E_k^+ \\ E_k^- \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} E_0^+ \\ E_0^- \end{pmatrix}$$

con

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \prod_{\nu=1}^{k} \begin{pmatrix} a_\nu & b_\nu \exp(-2i\alpha_\nu) \\ b_\nu \exp(2i\alpha_\nu) & a_\nu \end{pmatrix}$$

essendo $a_\nu, b_\nu = (1 \pm n_{\nu-1}/n_\nu)/2$ e $\alpha_\nu$ la fase all'inizio della lamina $\nu$. Ponendo $E_k^- = 0$, si trova il fattore di riflessione (del flusso) $r_0 = c_{12}c_{21}/c_{11}c_{22}$ e quello di trasmissione $t_k = 1 - r_0$. Il procedimento viene generalizzato al caso d'incidenza obliqua e di strati assorbenti e applicato ad alcuni casi particolari. Nel caso di un indice di rifrazione variabile con continuità l'autore trova con un procedimento di passaggio al limite

$$(1)\quad \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \exp \begin{Bmatrix} -\tfrac{1}{2}\log(n_k/n_0) & \tfrac{1}{2}\int_{n_0}^{n_k} \exp(-2i\alpha_n)\, d\log n \\ \tfrac{1}{2}\int_{n_0}^{n_k} \exp(2i\alpha_n)\, d\log n & -\tfrac{1}{2}\log(n_k/n_0) \end{Bmatrix}.$$

Questa matrice può essere valutata con uno sviluppo in serie. [Nota del revisore. Nel caso continuo l'autore pone $\alpha_n = (2\pi/\lambda)\int_0^x n\, dx$. Ma questa non è che una prima approssimazione (WKB). In generale la funzione $\alpha_n$, che compare nella (1), è incognita, e il metodo perde di valore pratico.]

*G. Toraldo di Francia* (Firenze).

**Pohlack, Hubert. Analytische und graphische Methoden zur Lösung optischer Interferenzprobleme bei dünnen Schichten.** Ann. Physik (6) **5**, 311–328 (1950).

Il calcolo delle matrici viene applicato ai problemi ottici dei sistemi di strati sottili, come già più volte è stato fatto negli ultimi anni. [L'autore stesso dichiara di non avere la possibilità di consultare la letteratura scientifica occidentale sull'argomento.] Viene anche suggerito il metodo grafico approssimato, che consiste nel rappresentare mediante un vettore nel piano il coefficiente di riflessione di ciascuna superficie, con l'opportuno ritardo di fase, ed eseguire poi la somma vettoriale. *G. Toraldo di Francia* (Firenze).

**Parke, Nathan Grier, III. Optical algebra.** J. Math. Physics **28**, 131–139 (1949).

This paper deals with the foundation of a statistical theory of light bearing the same relation to phenomenological optics that statistical mechanics bears to thermodynamics. Three primary theories are involved: those of N. Wiener [Acta Math. **55**, 117–258 (1930)], of R. C. Jones [J. Opt. Soc. Amer. **31**, 488–493 (1941); **38**, 671–685 (1948)], and of H. Mueller [lecture course and unpublished manuscript]. The author correlates the generalized harmonic analysis of Wiener with the frequency algebra of Jones (based on the electric vector) and the frequency algebra of Mueller (based on the Stokes vector). In doing this he has extended the theories of Jones and Mueller by creating time algebras corresponding to their frequency

algebras, also generalizing the frequency algebra of Mueller. Applications to scattering phenomena are given.

*J. L. Synge* (Dublin).

**Sobolev, V. V. On the problem of the diffuse reflection and transmission of light.** Doklady Akad. Nauk SSSR (N.S.) **69**, 547–550 (1949). (Russian)

V. A. Ambarzumian [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 229–232 (1943); these Rev. **5**, 52] expressed the coefficients of brightness in this problem in terms of certain auxiliary functions of a single variable. In this note the author obtains linear integral equations for these auxiliary functions. Exact solutions are obtained for the case of infinite optical depth and spherically symmetrical scattering, and a method of finding exact solutions is indicated for other scattering laws. In the case of finite optical depth and spherically symmetrical scattering the equations are transformed into a form suitable for numerical solution.

*F. Smithies* (Cambridge, England).

**Panyč, O. I. On the application of boundary conditions in diffraction problems.** Doklady Akad. Nauk SSSR (N.S.) **70**, 589–592 (1950). (Russian)

Il corpo diffrangente sia conduttore (ma non perfetto) ed occupi la regione chiusa $T_i$, limitata dalla superficie $S$; il resto dello spazio sia $T_e$. La funzione d'onda nel caso scalare soddisferà l'equazione $\Delta u+k_e^2u=-\rho$ in $T_e$ e $\Delta u+k_i^2u=0$ in $T_i$, mentre su $S$ dovrà essere $u_i=u_e$ e $\partial u_i/\partial n=\partial u_e/\partial n$. Se $G(P, Q)$ è la funzione di Green relativa al caso del conduttore perfetto ($G=0$ su $S$) e si pone

$$u_0(P)=\int_{T_e}G(P, Q)\rho(Q)d\tau_Q,$$

la funzione d'onda può scriversi nel modo seguente in $T_i$ e in $T_e$ rispettivamente:

$$u_i(P)=\int_S\mu(Q)\frac{e^{ik_iR(P,Q)}}{R(P, Q)}dS_Q,$$

$$u_e(P)=u_0(P)+\left[\int_S\frac{\partial G(P, Q')}{\partial n_{Q'}}dS_{Q'}\int_S\mu(Q)\frac{e^{ik_eR(Q',Q)}}{R(Q', Q)}dS_Q\right]$$

e soddisfa automaticamente la prima condizione al contorno. La seconda condizione fornisce un'equazione integrale per $\mu(Q)$. Approfittando del fatto che $|k_i|\gg|k_e|$, si possono sviluppare gl'integrali che compaiono in tale equazione in una serie di potenze di $1/k_i$. Così facendo, si trova che, a meno di infinitesimi di ordine superiore al terzo, la condizione approssimata di Leontovič, a cui deve soddisfare la funzione d'onda su $S$ è

$$u=-\frac{1}{ik_i}\left\{\frac{\partial u}{\partial n}\left[1-\frac{\bar\kappa}{ik_i}-\left(\frac{\bar\kappa}{k_i}\right)^2-\frac{(\kappa_1-\kappa_2)^2}{8k_i^2}\right]-\frac{1}{2k_i^2}\frac{\partial}{\partial n}\left(\frac{\partial^2u}{\partial l^2}+\frac{\partial^2u}{\partial m^2}\right)\right\}$$

dove $\kappa_1$ e $\kappa_2$ rappresentano le curvature principali di $S$ ($\kappa_1\geqq\kappa_2$), $\bar\kappa$ la loro media e $l$ e $m$ le coordinate curvilinee lungo le rispettive linee di curvatura. Con procedimento analogo si trovano nel caso elettromagnetico le condizioni su $S$ (esatte fino al secondo ordine)

$$E_l=-\frac{k_e\mu}{k_i}\left(1+\frac{\kappa_1-\kappa_2}{2ik_i}\right)H_m,\quad E_m=\frac{k_e\mu}{k_i}\left(1+\frac{\kappa_2-\kappa_1}{2ik_i}\right)H_l,$$

$$\frac{\partial E_n}{\partial n}+2\bar\kappa E_n=-\frac{ik_e^2\mu}{k_i}\left(1-\frac{\bar\kappa}{ik_i}\right)E_n.$$

*G. Toraldo di Francia* (Firenze).

**Ljunggren, T. Contributions to the theory of diffraction of electromagnetic waves by spherical particles. II.** Ark. Fys. **1**, 1–18 (1949).

Part I of the paper appeared in Ark. Mat. Astr. Fys. **36A**, no. 14 (1949) [these Rev. **11**, 294]. In Part II the author discusses the diffraction in the forward direction in connection with the determination of the size of spherical particles in mists and clouds by observations on the minima of intensity in the forward scattering. An error in Mecke's papers [Ann. Physik (4) **61**, 471–500 (1920); **62**, 623–648 (1920)] is pointed out. Reference is made to Köhler's work [Ark. Mat. Astr. Fys. **24A**, no. 9 (1934)]. Numerical results in the range of $ka$ from 20 to 60 are presented. A list of corrections to part I is included. *C. Bouwkamp.*

**Buchholz, Herbert. Die Ausbreitung elektromagnetischer Wellen im konzentrischen Breitbandkabel mit dielektrischen Längsstegen.** Arch. Elektr. Übertragung **1**, 137–150 (1947).

At very short wavelengths the dielectric inserts between the coaxial conductors of a broadband cable may have an appreciable effect on the field. To analyse wave propagation through the cable the author assumes that the cross-section may be represented by wedge-shaped sections of dielectric material arranged symmetrically about the axis. Then the cable is supposed cut to the center, and unrolled to form a wave guide, which consists of two parallel conducting planes (the inner and outer conductors) with alternate air and dielectric layers recurring cyclically between the planes. The usual wave guide technique is now used to determine the field between the planes. Assuming a longitudinal factor $\exp(i\gamma z)$ two fundamental solutions are obtained, defined by the condition that the transverse field component $E_y$ normal to the planes should be an even and odd function, respectively, of the tangential coordinate $x$ (assuming the origin to be chosen at the midpoint of one of the dielectric channels). In general each solution includes all six components of $E$ and $H$, but for the dominant mode in the first case there are only three nonvanishing field components, $E_y$, $H_x$ and $H_z$. This wave is discussed in detail, including determination of the critical frequencies and corresponding propagation constants, and plots of the magnetic field lines in the transverse plane. The additional heat losses in the conductors due to the dielectric inserts are also evaluated approximately. *M. C. Gray* (Murray Hill, N. J.).

**Tai, C. T. Application of a variational principle to biconical antennas.** J. Appl. Phys. **20**, 1076–1084 (1949).

The author first obtains an integral equation for the aperture field of a biconical antenna of length $l$ and angle $\theta_0$ (the aperture is that part of the boundary sphere $r=l$ which lies outside the cones), and thence derives an expression for the terminating admittance which is stationary with respect to small variations in the field. If we denote by $L_n(\theta)$ an orthogonal set of solutions of Legendre's equation which vanish at $\theta=\theta_0$, $\pi/2$, and $\pi-\theta_0$, an expansion for the aperture field, $E_a(\theta)=-\csc\theta+\sum A_nL_n'(\theta)$, may be assumed. Application of the variational method then leads to a set of linear equations to determine the coefficients $A_n$ and to a final expansion $Y_i=Y_{i0}+\sum_n\alpha_nA_n$ for the terminating admittance. The first approximation $Y_{i0}$ corresponds to the value of the terminating admittance obtained by neglecting all the complementary waves in the interior region, and agrees with the formula obtained by a different method by P. D. P. Smith [J. Appl. Phys. **19**, 11–23 (1948); these Rev. **9**, 552]. For cones of small angle the coefficients $A_n$ may be evaluated

explicitly, and Schelkunoff's original solution is thus verified independently. For cones of wide angle the author evaluates several terms of the admittance series for cone angles which correspond to odd integral values of the characteristic numbers $n$, and shows that the series converge satisfactorily. The paper concludes with a discussion of the evaluation of the characteristic numbers for arbitrary angles based on an application of the Ritz method. *M. C. Gray.*

**Mirimanov, R. G. Radiation resistance of a dipole located at the center of a thin spherical shell.** Doklady Akad. Nauk SSSR (N.S.) **71**, 1061–1064 (1950). (Russian)

The three media (inner sphere, shell, and outer region) have arbitrary electromagnetic parameters, none apparently being perfect conductors. From the exact general solution is derived a solution correct to the first power of the shell-thickness, assuming that the inner and outer media are the same. A complicated expression is then found for the mean energy-flow at large distances from the source. Apart from the latter result, the problem and approach are those of Keller and Keller [J. Appl. Phys. **20**, 393–396 (1949); these Rev. **10**, 659] who, however, get somewhat different results. [Reviewer's comments. The condensed exposition and numerous misprints make it difficult to check the results. The author works with a radial Hertz vector, and assumes that its divergence, i.e., the scalar potential with reversed sign, depends only on $r$. This would seem to lead to a null field; however, only one of the field-vectors is evaluated, and that is equated to a scalar quantity (equation (14)). No justification is quoted for the boundary conditions on the scalar potential, which are $\varphi|_i=\varphi|_j$, $\epsilon_i(\partial\varphi/\partial n)|_i=\epsilon_j(\partial\varphi/\partial n)|_j$, as in electrostatics.] *F. V. Atkinson* (Ibadan).

**Papas, Charles H. Radiation from a transverse slot in an infinite cylinder.** J. Math. Physics **28**, 227–236 (1950).

A transverse slot in an infinitely long perfectly conducting cylinder is fed by a rectangular waveguide (inside the cylinder) carrying a wave whose electric field has only one component, $E_z$, parallel to the axis of the cylinder. The author assumes that the field across the slot is approximately equal to the waveguide distribution $E_z$, and uses Kirchhoff's method to determine the radiated field. If the field in the slot is $E_z(a, \varphi', z')=Z(z')\Phi(\varphi')$ then the method of steepest descents leads to a final formula for the field at large distances:

$$E_z(\rho, \varphi, z)=\frac{1}{2\pi^2}\sum_{m=0}^{\infty}\frac{\epsilon_m \exp[-i(m+1)\pi/2]}{H_m^{(1)}(ka\cos\theta)}$$
$$\times\int_{-\infty}^{\infty}e^{-ikz'\sin\theta}Z(z')dz'\int_0^{2\pi}\Phi(\varphi')\cos m(\varphi-\varphi')d\varphi'.$$

The integrals are evaluated for a $TE_{01}$ wave across a narrow slot of height $w$ and width equal to $\lambda/2$. *M. C. Gray.*

**Brodin, Jean. Cas singulier du problème de Huyghens.** C. R. Acad. Sci. Paris **230**, 1345–1347 (1950).

**Brodin, Jean. Espace vectoriel des ondes régulières à l'extérieur d'une surface fermée.** C. R. Acad. Sci. Paris **230**, 1388–1390 (1950).

The electromagnetic field due to monochromatic sources inside a closed surface $S$ is expressed as the resultant of fields due to layers of electric and magnetic dipoles tangential to $S$. *E. T. Copson* (Dundee).

## *Quantum Mechanics*

**Wigner, Eugene P. Do the equations of motion determine the quantum mechanical commutation relations?** Physical Rev. (2) **77**, 711–712 (1950).

The author raises the question whether the postulate that the quantum mechanical operators obey the classical equations of motion uniquely determines the commutation rules. The answer is found to depend on the form of the Hamiltonian and is in the negative for a free particle and for the harmonic oscillator. *F. W. London.*

**Al'perin, M. M. On forced oscillations in quantum mechanics.** Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **20**, 62–64 (1950). (Russian)

The author considers a one-dimensional oscillator described by the Schrödinger equation

$$i\hbar\partial\Psi/\partial t=[H^\circ-qF(t)]\Psi,$$

where $H^\circ$ is the unperturbed Hamiltonian, $q$ the coordinate and $F(t)$ an arbitrary function of the time. Taking for $\Psi$ an expansion in terms of the eigenfunctions of $H^\circ$ with time-dependent coefficients, he obtains exact expressions for the coefficients. By means of these he investigates the transfer of energy between the oscillator and the external, or perturbing, system. *N. Rosen* (Chapel Hill, N. C.).

**Koba, Z. On the integrability condition of Tomonaga-Schwinger equation.** Progress Theoret. Physics **5**, 139–141 (1950).

The author discusses the normal-dependent terms which appear in the interaction Hamiltonian in the Tomonaga-Schwinger formalism of meson field theories with derivative interactions. He indicates some general arguments showing that such terms always disappear when the theories are applied to specific calculations using perturbation methods. The terms are exactly cancelled by second-order effects of the terms in the interaction containing derivatives.
*F. J. Dyson* (Princeton, N. J.).

**Koba, Zirô. Semi-classical treatment of the reactive corrections. I. The anomalous magnetic moment of the electron.** Progress Theoret. Physics **4**, 319–330 (1949).

Welton [Physical Rev. (2) **74**, 1157–1167 (1948)] has shown that the electromagnetic shift of energy levels can be ascribed to a coupling of the electron with the zero-point fluctuations of the electromagnetic field, but in applying this method to the calculation of the anomalous magnetic moment of the electron he gets the wrong sign. It is pointed out that the right correction to the magnetic moment of the electron results when one combines Welton's fluctuation with Schroedinger's "Zitterbewegung." Although the effect is essentially due to the possibility of transitions into negative energy states, a simple model can be used for its demonstration. If the electron is represented by a ring current, the fluctuations will reduce its effective magnetic moment [Welton], but the ring will also vibrate in its own plane, and this leads to an increase in the diameter of the ring and thus also to an increase in the effective magnetic moment, about twice as large as Welton's decrease. The correct value for the anomalous magnetic moment of the electron is thus obtained. *E. Gora.*

**Petiau, Gérard. Sur les équations d'ondes des corpuscules de spin $\frac{h}{4\pi}$ et leurs solutions dans les champs nucléaires généraux.** J. Phys. Radium (8) **10**, 264–276 (1949).